hep-th/0601101

Conserved Charges in Even Dimensional Asymptotically locally Anti-de Sitter Space-times

Rong-Gen Cai^{*,1} and Li-Ming Cao^{*,†,2}

* *Institute of Theoretical Physics, Chinese Academy of Sciences, P.O. Box 2735, Beijing 100080, China*

† *Graduate School of the Chinese Academy of Sciences, Beijing 100039, China*

ABSTRACT

Based on the recent paper hep-th/0503045, we derive a formula of calculating conserved charges in even dimensional asymptotically *locally* anti-de Sitter space-times by using the definition of Wald and Zoupas. This formula generalizes the one proposed by Ashtekar *et al.* Using the new formula we compute the masses of Taub-Bolt-AdS space-times by treating Taub-Nut-AdS space-times as the reference solution. Our result agrees with those resulting from "background subtraction" method or "boundary counterterm" method. We also calculate the conserved charges of Kerr-Taub-Nut-AdS solutions in four dimensions and higher dimensional Kerr-AdS solutions with Nut charges. The mass of (un)wrapped brane solutions in any dimension is given.

[1]Email address: cairg@itp.ac.cn
[2]Email address: caolm@itp.ac.cn

1 Introduction

Asymptotically AdS space-times have been investigated thoroughly in recent years due to the AdS/CFT correspondence [1, 2, 3]. It relates the gravity in asymptotic AdS space-times to dual conformal field theories (CFTs) living on the boundary of the AdS space-times. In this correspondence, the boundary fields which set the boundary conditions of bulk fields are identified with CFT sources which couple to gauge invariant operators. For example, the boundary metric plays the role of the metric of the space-time on which the dual field theory is defined; on the other hand, it is a source of the energy momentum tensor of boundary CFT. This requires the existence of space-times associated with general Dirichlet boundary conditions for the metric. Such general boundary conditions include the so-called asymptotically *locally* anti-de Sitter (AlAdS) space-times, which require only asymptotically *locally* AdS, not exact AdS. Asymptotically exact AdS case has been studied in the literatures, for example see [4, 5], and hereafter we call them AAdS space-times for simplicity. Note that for the n-dimensional AAdS space-times, the boundary has the topology $\mathbf{R} \times \mathbf{S}^{n-2}$, while for the AlAdS space-times, the topology of their boundary needs not to be $\mathbf{R} \times \mathbf{S}^{n-2}$. Due to the difference between the boundary topologies, some methods, which are very powerful to calculate conserved charges for AAdS space-times, do not work for AlAdS space-times.

Therefore, it is interesting to study the conserved charges in AlAdS space-times in its own right. In the literatures, there are different methods to obtain conserved charges for AAdS space-times, see for example, references [4, 5, 6, 7, 8, 9, 10, 11]. The comparison among these notions of conserved charges in AAdS space-times has been made by Hollands *et al.* in a recent paper [12]. For AAdS space-times, the method of Ashtekar *et al* and the method of "boundary counterterm" are independent of the reference background. Using these two methods, one therefore needs not consider this reference background problem and can get the conserved charges of AAdS space-times by straightforward calculations. However, for the AlAdS space-times, those methods which rigorously depend upon the boundary conditions of AAdS space-times would invalidate, because the boundary conditions of AlAdS space-times may be very different from those of AAdS space-times. For example, the method of Ashtekar *et al.* [5, 10] does not work for AlAdS space-times because the boundary topology $\mathbf{R} \times \mathbf{S}^{n-2}$ is required in this method. Moreover, the definition of n-dimensional AAdS space-times requires a condition that the product of the conformal factor to the power of $(3 - n)$ and the Weyl tensor of the unphysical space-time admits a smooth limit on the conformal boundary [10]. This condition can not be fulfilled for general AlAdS space-times. For example. one can easily check that the method of Ashtekar *et al.* does not work for the asymptotically AdS space-times with Nut charges, which are typical AlAdS space-times and will be discussed in the

present paper.

Several methods have been proposed by some authors to define the conserved charges in AlAdS space-times, such as the "holographic charges" studied in [13], the method given in [14, 15], and the superpotential method [16]. In this paper, based on the recent work of Hollands *et al* [12], we develop a method of calculating conserved charges in even dimensional AlAdS space-times by using the covariant phase space definition of Wald and Zoupas [17]. This method generalizes the formula of Ashtekar *et al.* such that we can calculate conserved charges in even dimensional AlAdS space-times. This method is background dependent, and one has to specify a reference background before calculating the conserved charges for these AlAdS space-times.

This paper is organized as follows. In the next section, we briefly review the definition of conserved charges given by Wald and Zoupas [17], and give explicit forms of some related quantities in AlAdS space-times. In Sec. 3, we derive the formula of calculating conserved charges in even dimensional AlAdS space-times by using the analysis of Hollands *et al.* [12]. Our formula generalizes the one given by Ashtekar *et al.*. In Sec. 4, using this new formula, we calculate the masses of Taub-Bolt-AdS solutions by treating the Taub-Nut-AdS space-times as the reference background. In Sec. 5, four dimensional Kerr-Taub-Nut-AdS solution is discussed, and the mass and angular momentum associated to it are calculated. Sec. 6 is devoted to calculating the conserved charges for higher dimensional Kerr-AdS solutions with Nut charges. In Sec. 7 we give the mass for (un)wrapped black brane solutions. We end in Sec. 8 with conclusion and discussion.

2 Wald's Definition and the Boundary of AlAdS Space-times

In differential covariant theories of gravity, Wald *et al.*[17] developed a general prescription to define "conserved charges" at asymptotic boundaries for any space-times. In this paper, we will use this method to define the conserved charges dual to some asymptotic symmetry generators of AlAdS space-times. For simplicity, we consider no matter case (i.e., g_{ab} are the only dynamical fields), therefore the corresponding differential covariant Lagrangian of n-dimensional AlAdS space-times (M, g_{ab}) is:

$$\mathbf{L} = \frac{1}{16\pi G}\left[R - 2\Lambda\right]\boldsymbol{\epsilon}, \tag{1}$$

where we have put the Lagrangian in the form of differential form and $\boldsymbol{\epsilon}$ is the volume element. The variation of the Lagrange density $\mathbf{L}$ can be written as

$$\delta\mathbf{L} = \mathbf{E}^{ab}\delta g_{ab} + d\boldsymbol{\Theta}, \tag{2}$$

where $\mathbf{\Theta}$ is an $(n-1)$-form, which is called *symplectic potential form*, and it is a local linear function of field variation. $\mathbf{E}^{ab}$ corresponds to the equations of motion. Their explicit forms are

$$\mathbf{\Theta}_{a_1\cdots a_{n-1}}(g_{ab},\delta g_{ab}) = \frac{1}{16\pi G}\boldsymbol{\epsilon}_{a_1\cdots a_{n-1}a}\boldsymbol{v}^a(g_{ab},\delta g_{ab}), \tag{3}$$

$$\mathbf{E}^{ab} = \frac{1}{16\pi G}\left[R^{ab} - \frac{1}{2}Rg^{ab} + \Lambda g^{ab}\right]\boldsymbol{\epsilon}, \tag{4}$$

where

$$\boldsymbol{v}^a = g^{ab}\nabla^c\delta g_{cb} - g^{bc}\nabla^a\delta g_{bc}. \tag{5}$$

The *symplectic current* $(n-1)-$*form* $\boldsymbol{\omega}$ is defined by taking an antisymmetrized variation of $\mathbf{\Theta}$:

$$\boldsymbol{\omega}_{a_1\cdots a_{n-1}}(g_{ab},\delta_1 g_{ab},\delta_2 g_{ab}) = \frac{1}{16\pi G}\boldsymbol{\epsilon}_{a_1\cdots a_{n-1}a}\boldsymbol{w}^a(g_{ab},\delta_1 g_{ab},\delta_2 g_{ab}), \tag{6}$$

where $\boldsymbol{w}$ is a 1-form given by

$$\boldsymbol{w}^a(g_{ab},\delta_1 g_{ab},\delta_2 g_{ab}) = P^{abcdef}(\delta_1 g_{bc}\nabla_d\delta_2 g_{ef} - \delta_2 g_{bc}\nabla_d\delta_1 g_{ef}), \tag{7}$$

where

$$P^{abcdef} = g^{ae}g^{fb}g^{cd} - \frac{1}{2}g^{ad}g^{be}g^{fc} - \frac{1}{2}g^{ab}g^{cd}g^{ef} - \frac{1}{2}g^{bc}g^{ae}g^{fd} + \frac{1}{2}g^{bc}g^{ad}g^{ef}. \tag{8}$$

The integral of the symplectic current form over an $(n-1)-$dimensional submanifold Σ on $\widetilde{M}$ gives the *presymplectic form*,

$$\Omega_\Sigma(g,\delta_1 g,\delta_2 g) = \int_\Sigma \boldsymbol{\omega}(g,\delta_1 g,\delta_2 g), \tag{9}$$

where $\widetilde{M} = M\cup\mathcal{B}$ is the conformal completion of M with the boundary manifold $\mathcal{B}$. The presymplectic structure Ω_Σ does not depend on the choice of Σ if $\delta_1 g$ and $\delta_2 g$ satisfy linearized field equations and g has suitable asymptotic condition [17, 18, 19, 20]. Here we have assumed that "kinetically" allowed field space $\mathcal{F}$ has been defined such that $\boldsymbol{\omega}$ can be extended continuously to $\mathcal{B}$ for all $\delta_1 g$ and $\delta_2 g$ tangent to the solution subspace $\overline{\mathcal{F}}$ and Σ has an unambiguous boundary $\partial\Sigma$ in $\mathcal{B}$.

The "conserved charges" $H_\xi : \overline{\mathcal{F}} \to \mathbf{R}$ associated with a vector field ξ^a representing an asymptotic symmetry defined by using presymplectic form in [17] satisfies

$$\delta H_\xi = \Omega_\Sigma(g;\delta g,\mathcal{L}_\xi g), \tag{10}$$

for an arbitrary δg which is tangent to field space $\mathcal{F}$ at point g of the solution subspace $\overline{\mathcal{F}}$. One can put it in the form [20]

$$\delta H_\xi = \int_\Sigma \xi^a\delta\mathbf{C}_a + \int_{\partial\Sigma}[\delta\mathbf{Q} - \xi\cdot\mathbf{\Theta}], \tag{11}$$

where

$$\mathbf{Q}_{a_1\ldots a_{n-2}} = -\frac{1}{16\pi G}(\nabla^b\xi^c)\boldsymbol{\epsilon}_{bca_1\ldots a_{n-2}}. \tag{12}$$

If the equations of motion hold, then $\mathbf{C}_a = 0$, i.e., $\mathbf{C}_a$ correspond to "constraints" of the theory. Equation (12) defines the *Noether charge* $(n-2)$*-form*, $\mathbf{Q}$. If δg is tangent to $\overline{\mathcal{F}}$, or satisfies linearized field equations, then (11) becomes

$$\delta H_\xi = \int_{\partial\Sigma}[\delta\mathbf{Q} - \xi\cdot\mathbf{\Theta}]. \tag{13}$$

It was shown in [17] that if $\boldsymbol{\omega} = 0$ on $\mathcal{B}$ (case I), or ξ is everywhere tangent to the cross section $\partial\Sigma$ in $\mathcal{B}$ (case II), then "conserved charges" H_ξ exist, and in case I, they are really conserved.

It should be noted, in one connected component of $\overline{\mathcal{F}}$, that the conserved charge (10) is uniquely defined up to a constant. Usually, we can choose a natural "reference solution" $g_0 \in \overline{\mathcal{F}}$ so that this constant, $H_\xi[g_0]$, vanishes. Integrating the variation parameter from 0 to $\bar\lambda$ (which corresponds to the solution we want to define the conserved charges.), the conserved charge $H_\xi[g_{\bar\lambda}]$ is given by

$$H_\xi[g_{\bar\lambda}] = \int_0^{\bar\lambda} d\lambda \int_{\partial\Sigma}[\delta\mathbf{Q}_\lambda - \xi\cdot\mathbf{\Theta}_\lambda]. \tag{14}$$

Just as pointed out by the authors of the paper [17], this definition does not depend on the choice of the paths connecting g_0 and $g_{\bar\lambda}$. So the conserved charges are well defined. In the next section, we will use the definition and give the conserved charges in AlAdS space-times.

In the remainder of this section, we will give some preliminary analysis about the neighborhood of boundary of the AlAdS space-times. The n-dimensional AlAdS space-times are solutions of Einstein's equations with a negative cosmological constant, whose Riemann tensor asymptotically approaches to that of exact AdS space-time. A simple class of AlAdS space-times is AAdS space-times which have boundary topology $\mathbf{R}\times\mathbf{S}^{n-2}$. For general AlAdS space-times their boundary topology may be different from the topology $\mathbf{R}\times\mathbf{S}^{n-2}$.

Let $(M, g_\lambda), \lambda \in \mathbf{R}$, be a smooth one-parameter family of n-dimensional AlAdS space-times which pass through the point $g_{\bar\lambda}$ (which corresponds to the AlAdS space-time under discussion) in the solution subspace $\overline{\Gamma}$. We assume that they have the same conformal infinity. That is to say (i) one can attach a boundary $\mathcal{B}$ to M such that $\widetilde{M} = M\cup\mathcal{B}$ is a manifold with boundary. For example $\mathcal{B}\cong\mathbf{R}\times\Im^{n-2}$ and $\Im^{n-2}$ denotes an $(n-2)$-dimensional manifold whose topology may not be that of a round sphere $\mathbf{S}^{n-2}$; (ii) on $\widetilde{M}$, there is a family of smooth metrics $\bar g_\lambda$ and a smooth function Ω (does not depend λ) such that $g_\lambda = \Omega^{-2}\bar g_\lambda$, and such that $\Omega = 0$,

$$d\Omega \neq 0, \tag{15}$$

at points of $\mathcal{B}$. The metrics on $\mathcal{B}$ induced by $\bar{g}_\lambda$ are of the same form for all λ, and can be denoted as h. For example, for the Schwarzschild-AdS black holes with mass parameter as a variation parameter, their metrics can be expressed as

$$ds^2_\lambda = -(k - \frac{2\lambda}{r^{n-3}} + \frac{r^2}{\ell^2})dt^2 + \frac{dr^2}{k - \frac{2\lambda}{r^{n-3}} + \frac{r^2}{\ell^2}} + r^2 d\sigma^2_{n-2}, \tag{16}$$

where $d\sigma^2_{n-2}$ denotes for the metric of $(n-2)$-dimensional Einstein space with constant curvature $(n-2)(n-3)k$. In the case of $k=1$, one can choose a conformal factor and boundary as $\Omega = \frac{\ell}{r}$, $\mathcal{B} = \mathbf{R} \times \mathbf{S}^{n-2}$, and realize the completion described above. The boundary metric h is an Einstein static unverse with radius ℓ

$$d\bar{s}^2_\lambda|_{\mathcal{B}} = \Omega^2 ds^2_\lambda|_{\mathcal{B}} = -dt^2 + \ell^2 d\sigma^2_{n-2}. \tag{17}$$

For h and each λ, in the neighborhood of $\mathcal{B}$, there exists a unique conformal factor ρ_λ or coordinates $x_\lambda = (\rho_\lambda, y)$ in which the metric takes the form [21, 22]

$$\begin{aligned} \rho^2_\lambda g_\lambda &= \widetilde{g}_\lambda = d\rho^2_\lambda + \widetilde{h}_{\rho_\lambda} \\ \widetilde{h}_{\rho_\lambda} &= \widetilde{h}_0 + \rho_\lambda (\widetilde{h}_\lambda)_1 + \cdots + \rho^{n-1}_\lambda (\widetilde{h}_\lambda)_{n-1} + (\alpha_\lambda)_{n-1} \rho^{n-1}_\lambda \ln \rho^2_\lambda + \cdots \end{aligned} \tag{18}$$

where $\widetilde{h}_{\rho_\lambda}$ is chosen such that $\widetilde{h}_{\rho_\lambda=0}$ is equal to the metric $\widetilde{h}_0 = h$ on $\mathcal{B}$, and in the neighborhood of the boundary $\mathcal{B}$,

$$(\widetilde{h}_{\rho_\lambda})_{ab} {}^{(\lambda)}\widetilde{\nabla}^b \rho_\lambda = 0, \quad (\widetilde{g}_\lambda)^{ab} {}^{(\lambda)}\widetilde{\nabla}_a \rho_\lambda {}^{(\lambda)}\widetilde{\nabla}_b \rho_\lambda = 1, \tag{19}$$

where ${}^{(\lambda)}\widetilde{\nabla}_a$ is the covariant derivative associated to $\widetilde{g}_\lambda$. $(\widetilde{h}_{\rho_\lambda})_{ab}$ is the induced metric on the surfaces $\mathcal{B}_{\rho_\lambda}$, the time-like surfaces of constant ρ_λ with coordinate y. In fact, ρ_λ is just the distance from the point to the boundary. Here, for simplicity, we have set the AdS radius $\ell = 1$.

Straightforward computation shows that the Riemann tensor of (18) is of the form of exact AdS up to a correction of order ρ^{-3}_λ [21, 23]. The asymptotic analysis reveals that all coefficients shown in (18) except the traceless and divergenceless part of $(\widetilde{h}_\lambda)_{n-1}$ are locally determined in terms of boundary data. So, $(\widetilde{h}_\lambda)_j$, for $j \leq n-2$ are independent of λ. The logarithmic term appears only when n is odd, if one considers the pure gravity case. This term is important in the context of AdS/CFT in odd dimensional AlAdS space-times, which reflects the anomaly in the even-dimensional dual conformal field theories [24].

Assume that the coordinates of point $p \in \epsilon \times \mathcal{B}$ (The neighborhood of $\mathcal{B}$ has a direct product form, we denote it by $\epsilon \times \mathcal{B}$, where ϵ is a small quantity.) are $x_\lambda(p) = (\rho_\lambda(p), y(p))$. Consider differomorphism ϕ_σ of $\widetilde{M}$ which has a restriction on the neighborhood of $\mathcal{B}$

$$\phi_\sigma : \epsilon \times \mathcal{B} \rightarrow \epsilon \times \mathcal{B}, \quad p \mapsto \phi_\sigma(p) \tag{20}$$

with $\phi_\sigma(p)$ satisfying

$$x_\lambda(\phi_\sigma(p)) = (\rho_{\lambda+\sigma}(p), y(p)), \qquad \forall\ \lambda \in \mathbf{R} \tag{21}$$

where ϵ is small enough such that these coordinates are well defined in $\epsilon \times \mathcal{B}$ for all λ. Then ϕ_σ forms a one parameter transformation on $\epsilon \times \mathcal{B}$. If the vector field which generates ϕ_σ is denoted by ζ, then we have

$$\begin{aligned}
\mathcal{L}_\zeta \widetilde{g}_\lambda(p) &= \lim_{\sigma\to 0} \frac{1}{\sigma} \left[(\phi_\sigma^* \widetilde{g}_\lambda) - \widetilde{g}_\lambda\right](p) \\
&= \lim_{\sigma\to 0} \frac{1}{\sigma} \left[\phi_\sigma^*(d\rho_\lambda^2 + \widetilde{h}_{\rho_\lambda}) - (d\rho_\lambda^2 + \widetilde{h}_{\rho_\lambda})\right](p) \\
&= \lim_{\sigma\to 0} \frac{1}{\sigma} \left[\left(d(\rho_\lambda \circ \phi_\sigma)^2 + \phi_\sigma^* \widetilde{h}_{\rho_\lambda}\right) - \left(d\rho_\lambda^2 + \widetilde{h}_{\rho_\lambda}\right)\right](p) \\
&= \lim_{\sigma\to 0} \frac{1}{\sigma} \left[\left(d\rho_{\lambda+\sigma}^2 + \widetilde{h}_{\rho_{\lambda+\sigma}}\right) - \left(d\rho_\lambda^2 + \widetilde{h}_{\rho_\lambda}\right)\right](p) \\
&= \left[2d\left(\frac{d\rho_\lambda}{d\lambda}\right) d\rho_\lambda + \frac{\partial \widetilde{h}_{\rho_\lambda}}{\partial \rho_\lambda} \frac{d\rho_\lambda}{d\lambda}\right](p)
\end{aligned}$$

Thus, we have

$$\mathcal{L}_\zeta \widetilde{g}_\lambda = 2d\left(\frac{d\rho_\lambda}{d\lambda}\right) d\rho_\lambda + \frac{\partial \widetilde{h}_{\rho_\lambda}}{\partial \rho_\lambda} \frac{d\rho_\lambda}{d\lambda}. \tag{22}$$

It should be noted that the restriction of ϕ_σ on $\mathcal{B}$ is an identity. So, the vector field ζ must vanish on $\mathcal{B}$ if it generates ϕ_σ in $\epsilon \times \mathcal{B}$, i.e., it corresponds to a gauge freedom according to the asymptotic symmetry transformations. Consider the variation of $\widetilde{g}_\lambda$ at $\bar{\lambda}$

$$\frac{d}{d\lambda}\widetilde{g}_\lambda|_{\bar{\lambda}} = \frac{d}{d\lambda}(d\rho_\lambda^2 + \widetilde{h}_{\rho_\lambda})|_{\bar{\lambda}}. \tag{23}$$

With the help of (18) and (22), in the even dimensional case, one has

$$\begin{aligned}
\frac{d}{d\lambda}\widetilde{g}_\lambda|_{\bar{\lambda}} &= \left[2d\left(\frac{d\rho_\lambda}{d\lambda}\right) d\rho_\lambda + \frac{\partial \widetilde{h}_{\rho_\lambda}}{\partial \rho_\lambda} \frac{d\rho_\lambda}{d\lambda}\right]_{\bar{\lambda}} + \rho_{\bar{\lambda}}^{n-1} \frac{d}{d\lambda}(\widetilde{h}_\lambda)_{n-1}|_{\bar{\lambda}} + O(\rho_{\bar{\lambda}}^{n-1}) \\
&= \mathcal{L}_\zeta \widetilde{g}_{\bar{\lambda}} + \rho_{\bar{\lambda}}^{n-1} \frac{d}{d\lambda}(\widetilde{h}_\lambda)_{n-1}|_{\bar{\lambda}} + O(\rho_{\bar{\lambda}}^{n-1}).
\end{aligned} \tag{24}$$

Thus, if the change of ρ_λ with λ gives a contribution to the variation of $g_{\bar{\lambda}}$, then this contribution is just a Lie-derivative of $g_{\bar{\lambda}}$ about one vector field which vanishes on the boundary $\mathcal{B}$. This equation is very useful in our analysis below. The same derivation will be done for giving the variation of the electric part of Weyl tensor in the next section.

3 Conserved Charges in Even Dimensional AlAdS Space-times

To give the explicit form of conserved charges in AlAdS space-times, we need to analyze the field equations. Our analysis here is similar to that of Hollands *et al.* [12]. The difference is that we are

treating even dimensional AlAdS space-times instead of the AAdS space-times considered in [12]. Therefore the result is modified so that we can calculate the conserved charges of more general solutions to which the method of Ashtekar *et al* [10] is not applicable. The reader who wants to know more details of this procedure may refer to the paper [12].

3.1 Analysis of Einstein Equations

To analyze Einstein's equations, we follow [12] and introduce the tensor field for each λ in the AlAdS solution family $g_{\lambda,ab}$ described in the previous section

$$(\widetilde{S}_\lambda)_{ab} = \frac{2}{n-2}(\widetilde{R}_\lambda)_{ab} - \frac{1}{(n-1)(n-2)}\widetilde{R}_\lambda(\widetilde{g}_\lambda)_{ab}, \tag{25}$$

In terms of this field and the coordinates of (18), in the neighborhood of $\mathcal{B}$, Einstein's equations can be rewritten as

$$(\widetilde{S}_\lambda)_{ab} = -2\rho_\lambda^{-1}\, {}^{(\lambda)}\widetilde{\nabla}_a(\widetilde{n}_\lambda)_b. \tag{26}$$

Treating ρ_λ as "time", one can obtain the equations of "constraint" and "evolution" by using standard ADM decomposition. The constraint equations are

$$\begin{aligned} -\widetilde{\mathcal{R}}_\lambda - (\widetilde{K}_\lambda)_{ab}(\widetilde{K}_\lambda)^{ab} + \widetilde{K}_\lambda^2 + 2(n-2)\rho_\lambda^{-1}\widetilde{K}_\lambda &= 0\,, \qquad (27)\\ {}^{(\lambda)}\widetilde{D}^a(\widetilde{K}_\lambda)_{ab} - {}^{(\lambda)}\widetilde{D}_b\widetilde{K}_\lambda &= 0\,, \qquad (28)\end{aligned}$$

where ${}^{(\lambda)}\widetilde{D}_a$ is the derivative operator associated with $(\widetilde{h}_\lambda)_{ab}$, $(\widetilde{K}_\lambda)_{ab} = -(\widetilde{h}_\lambda)_a{}^c(\widetilde{h}_\lambda)_b{}^d\, {}^{(\lambda)}\widetilde{\nabla}_c(\widetilde{n}_\lambda)_d$ is the extrinsic curvature of the surfaces $\mathcal{B}_{\rho_\lambda}$ (with respect to the unphysical metric). Here we have denoted $(\widetilde{h}_{\rho_\lambda})_{ab}$ by $(\widetilde{h}_\lambda)_{ab}$ for simplicity, and $\widetilde{\mathcal{R}}_\lambda$ is the intrinsic Ricci scalar of $\mathcal{B}_{\rho_\lambda}$. The evolution equations are

$$\begin{aligned} \frac{d}{d\rho_\lambda}(\widetilde{K}_\lambda)_a{}^b &= (\widetilde{\mathcal{R}}_\lambda)_a{}^b + \widetilde{K}_\lambda(\widetilde{K}_\lambda)_a{}^b + \rho_\lambda^{-1}(n-2)(\widetilde{K}_\lambda)_a{}^b + \rho_\lambda^{-1}\widetilde{K}_\lambda\delta_a{}^b, \qquad (29)\\ \frac{d}{d\rho_\lambda}(\widetilde{h}_\lambda)_{ab} &= -2(\widetilde{h}_\lambda)_{bc}(\widetilde{K}_\lambda)_a{}^c. \qquad (30)\end{aligned}$$

By assumption, $\mathcal{B}$ is a smooth boundary, which implies that the fields $(\widetilde{h}_\lambda)_{ab}$ and $(\widetilde{K}_\lambda)_{ab}$ must be smooth in a neighborhood of $\mathcal{B}$. Consequently, multiplying the first evolution equation by ρ_λ and evaluating on $\mathcal{B}$, one can immediately get

$$(\widetilde{K}_\lambda)_{ab}|_{\mathcal{B}} = 0 = \frac{d}{d\rho_\lambda}(\widetilde{h}_\lambda)_{ab}|_{\mathcal{B}}. \tag{31}$$

To investigate more systematically the consequences implied by Eq. (29) and (30), we express them in terms of the traceless part $(\widetilde{p}_\lambda)_a{}^b$ of $(\widetilde{K}_\lambda)_a{}^b$ and use the familiar technique–Fefferman-Graham expansion [25]:

$$(\widetilde{h}_\lambda)_{ab} = \left(\widetilde{h}_{ab}\right)_0 + \rho_\lambda\left((\widetilde{h}_\lambda)_{ab}\right)_1 + \cdots + \rho_\lambda^{n-1}\left((\widetilde{h}_\lambda)_{ab}\right)_{n-1} + \rho_\lambda^{n}\left((\widetilde{h}_\lambda)_{ab}\right)_n + \cdots,$$

$$(\widetilde{p}_\lambda)_a{}^b = \left((\widetilde{p}_\lambda)_a{}^b\right)_0 + \rho_\lambda \left((\widetilde{p}_\lambda)_a{}^b\right)_1 + \cdots + \rho_\lambda^{n-1}\left((\widetilde{p}_\lambda)_a{}^b\right)_{n-1} + \rho_\lambda^n \left((\widetilde{p}_\lambda)_a{}^b\right)_n + \cdots . \tag{32}$$

The logarithmic terms have not been included because we consider even dimensional cases only, where each tensor $\left((\widetilde{h}_\lambda)_{ab}\right)_j$, $\left((\widetilde{p}_\lambda)_a{}^b\right)_j$ are independent of ρ_λ in the sense that the Lie-derivative along $(\widetilde{n}_\lambda)^a$ vanishes. Substituting the above expansion into (29) and (30), one can obtain the following recursion relations

$$\begin{aligned}(n-2-j)\left((\widetilde{p}_\lambda)_a{}^b\right)_j &= \left((\widetilde{\mathcal{R}}_\lambda)_a{}^b\right)_{j-1} - \frac{1}{n-1}\left(\widetilde{\mathcal{R}}_\lambda\right)_{j-1}\delta_a{}^b \\ &\quad - \sum_{m=0}^{j-1}\left(\widetilde{K}_\lambda\right)_m \left((\widetilde{p}_\lambda)_a{}^b\right)_{j-1-m}, \end{aligned} \tag{33}$$

$$(2n-3-j)\left(\widetilde{K}_\lambda\right)_j = \left(\widetilde{\mathcal{R}}_\lambda\right)_{j-1} - \sum_{m=0}^{j-1}\left(\widetilde{K}_\lambda\right)_m\left(\widetilde{K}_\lambda\right)_{j-1-m}, \tag{34}$$

and

$$j\left((\widetilde{h}_\lambda)_{ab}\right)_j = -2\sum_{m=0}^{j-1}\left[\left((\widetilde{h}_\lambda)_{bc}\right)_m\left((\widetilde{p}_\lambda)_a{}^b\right)_{j-1-m} + \frac{1}{n-1}\left((\widetilde{h}_\lambda)_{ab}\right)_m\left(\widetilde{K}_\lambda\right)_{j-1-m}\right]. \tag{35}$$

The "initial conditions" are, from Eq. (31),

$$\left((\widetilde{p}_\lambda)_a{}^b\right)_0 = \left(\widetilde{K}_\lambda\right)_0 = 0, \tag{36}$$

and $(\widetilde{h}_{ab})_0 = h_{ab}$ is the metric of the boundary $\mathcal{B}$. The key point of these equations is that $\left((\widetilde{h}_\lambda)_{ab}\right)_j$ and $\left((\widetilde{K}_\lambda)_a{}^b\right)_l$ are uniquely determined in terms of the initial conditions for $j < n-1$ and $l < n-2$. Therefore they are independent of λ. Thus, we have

$$\frac{d}{d\lambda}(\widetilde{h}_\lambda)_{ab}|_{\bar{\lambda}} = \left(\frac{\partial(\widetilde{h}_\lambda)_{ab}}{\partial\rho_\lambda}\frac{d\rho_\lambda}{d\lambda}\right)_{\bar{\lambda}} + \rho_{\bar{\lambda}}^{n-1}\frac{d}{d\lambda}\left((\widetilde{h}_\lambda)_{ab}\right)_{n-1}|_{\bar{\lambda}} + O(\rho_{\bar{\lambda}}^n). \tag{37}$$

As a result, any quantity that depends only on $\left((\widetilde{h}_\lambda)_{ab}\right)_j$ and $\left((\widetilde{K}_\lambda)_a{}^b\right)_l$ in range $j < n-1$ and $l < n-2$, must be automatically independent of λ.

The analysis of the recursion relation (33) for $j = n-2$ and constraint equation tell us that [12], once the traceless, symmetric tensor $\left((\widetilde{p}_\lambda)_a{}^b\right)_{n-2}$ with the divergence (determined by the constraint equations) is given, all tensors $\left((\widetilde{p}_\lambda)_a{}^b\right)_j$ and $\left((\widetilde{h}_\lambda)_{ab}\right)_j$ are uniquely determined for $j \geq n-1$ via the evolution and constraint equations. Thus, this tensor carries the full information about the metric $(\widetilde{g}_\lambda)_{ab}$ which is not already supplied by the boundary conditions, i.e., the "non-kinematical" information. The tensor $\left((\widetilde{p}_\lambda)_a{}^b\right)_{n-2}$ is related to the electric part of the unphysical Weyl tensor, as we will show shortly. From the definition of the tensor field $(\widetilde{S}_\lambda)_{ab}$, we have

$$(\widetilde{R}_\lambda)_{abcd} = (\widetilde{C}_\lambda)_{abcd} + (\widetilde{g}_\lambda)_{a[c}(\widetilde{S}_\lambda)_{d]b} - (\widetilde{g}_\lambda)_{b[c}(\widetilde{S}_\lambda)_{d]a}\,. \tag{38}$$

Using Einstein's equations, the definition of extrinsic curvature and the Gauss-Coddazi relation, we have

$$(\widetilde{C}_\lambda)^a{}_{bcd}(\widetilde{n}_\lambda)^b(\widetilde{n}_\lambda)^d = -(\widetilde{K}_\lambda)^a{}_b(\widetilde{K}_\lambda)^b{}_c + \mathcal{L}_{\widetilde{n}_\lambda}(\widetilde{K}_\lambda)^a{}_c - \rho_\lambda^{-1}(\widetilde{K}_\lambda)^a{}_c\,. \tag{39}$$

This equation can be expanded in powers of ρ_λ like the metric. We thereby obtain equations for the expansion coefficients. At the order $n-3$, one has the relation

$$\frac{1}{n-3}\left((\widetilde{C}_\lambda)_{acbd}(\widetilde{n}_\lambda)^c(\widetilde{n}_\lambda)^d\right)_{n-3} = \left((\widetilde{K}_\lambda)_{ab}\right)_{n-2} - \frac{1}{n-3}\sum_{m=0}^{n-3}\left((\widetilde{K}_\lambda)_{ac}\right)_m\left((\widetilde{K}_\lambda)_b{}^c\right)_{n-3-m}. \tag{40}$$

However, the coefficients appearing in the sum are independent of λ, and they are determined by boundary data. Consequently, we can obtain the variation relation

$$\frac{d}{d\lambda}\left[\frac{1}{n-3}\left((\widetilde{C}_\lambda)_{acbd}(\widetilde{n}_\lambda)^c(\widetilde{n}_\lambda)^d\right)_{n-3}\right]_{\bar\lambda} = \frac{d}{d\lambda}\left[\left((\widetilde{K}_\lambda)_{ab}\right)_{n-2}\right]_{\bar\lambda}. \tag{41}$$

It is easy to see, at order $j < n-3$, the coefficients of $(\widetilde{C}_\lambda)_{abcd}(\widetilde{n}_\lambda)^b(\widetilde{n}_\lambda)^d$ are independent of λ because they are fixed by the boundary data as we mentioned above. Thus, the variation of these coefficients are zero, i.e.,

$$\frac{d}{d\lambda}\left[\left((\widetilde{C}_\lambda)_{acbd}(\widetilde{n}_\lambda)^c(\widetilde{n}_\lambda)^d\right)_j\right]_{\bar\lambda} = 0. \tag{42}$$

Combining (35) with (41), we get

$$\frac{d}{d\lambda}\left[\frac{1}{n-3}\left((\widetilde{C}_\lambda)_{acbd}(\widetilde{n}_\lambda)^c(\widetilde{n}_\lambda)^d\right)_{n-3}\right]_{\bar\lambda} = \frac{d}{d\lambda}\left[-\frac{n-1}{2}\left((\widetilde{h}_\lambda)_{ab}\right)_{n-1}\right]_{\bar\lambda}. \tag{43}$$

Substituting this result into (37), we immediately have the following equation

$$\frac{d}{d\lambda}(\widetilde{h}_\lambda)_{ab}|_{\bar\lambda} = \left[\frac{\partial(\widetilde{h}_\lambda)_{ab}}{\partial\rho_\lambda}\frac{d\rho_\lambda}{d\lambda}\right]_{\bar\lambda} - \frac{2}{n-1}\rho_{\bar\lambda}^{n-1}\frac{d}{d\lambda}\left[\frac{1}{n-3}\left((\widetilde{C}_\lambda)_{acbd}(\widetilde{n}_\lambda)^c(\widetilde{n}_\lambda)^d\right)_{n-3}\right]_{\bar\lambda} + O(\rho_{\bar\lambda}^n). \tag{44}$$

The similar procedure to deduce Eq. (24) can be used to $(\widetilde{C}_\lambda)_{acbd}(\widetilde{n}_\lambda)^c(\widetilde{n}_\lambda)^d$, and once again one gets that the variation of $(\widetilde{C}_\lambda)_{acbd}(\widetilde{n}_\lambda)^c(\widetilde{n}_\lambda)^d$ can be divided into two parts (In fact, any quantity which can be expanded as (32) always has such a variation relation)

$$\begin{aligned}\frac{d}{d\lambda}\left[(\widetilde{C}_\lambda)_{acbd}(\widetilde{n}_\lambda)^c(\widetilde{n}_\lambda)^d\right]_{\bar\lambda} &= \mathcal{L}_\zeta\left[(\widetilde{C}_{\bar\lambda})_{abcd}(\widetilde{n}_{\bar\lambda})^b(\widetilde{n}_{\bar\lambda})^d\right] \\ &\quad +\rho_{\bar\lambda}^{n-3}\frac{d}{d\lambda}\left[\left((\widetilde{C}_\lambda)_{abcd}(\widetilde{n}_\lambda)^b(\widetilde{n}_\lambda)^d\right)_{n-3}\right]_{\bar\lambda} + O(\rho_{\bar\lambda}^{n-2}).\end{aligned} \tag{45}$$

Multiplying this equation by $\rho_{\bar\lambda}^{3-n}$ and considering $\mathcal{L}_\zeta\rho_{\bar\lambda} = \frac{d\rho_\lambda}{d\lambda}|_{\bar\lambda}$, we arrive at

$$\begin{aligned}\frac{d}{d\lambda}\left[\rho_\lambda^{3-n}(\widetilde{C}_\lambda)_{acbd}(\widetilde{n}_\lambda)^c(\widetilde{n}_\lambda)^d\right]_{\bar\lambda} &= \mathcal{L}_\zeta\left[\rho_{\bar\lambda}^{3-n}(\widetilde{C}_{\bar\lambda})_{acbd}(\widetilde{n}_{\bar\lambda})^c(\widetilde{n}_{\bar\lambda})^d\right] \\ &\quad +\frac{d}{d\lambda}\left[\left((\widetilde{C}_\lambda)_{acbd}(\widetilde{n}_\lambda)^c(\widetilde{n}_\lambda)^d\right)_{n-3}\right]_{\bar\lambda} + O(\rho_{\bar\lambda}).\end{aligned} \tag{46}$$

Define the electric part of the unphysical Weyl tensor as

$$(\widetilde{E}_\lambda)_{ab} = \frac{1}{n-3}\rho_\lambda^{3-n}\left((\widetilde{C}_\lambda)_{acbd}(\widetilde{n}_\lambda)^c(\widetilde{n}_\lambda)^d\right). \tag{47}$$

It should be noted here, although $(\widetilde{E}_\lambda)_{ab}$ may be divergent when ρ_λ approaches to zero, the variation $\frac{d}{d\lambda}(\widetilde{E}_\lambda)_{ab}$ is always finite if one fixes the conformal factor as a gauge condition, which can be understood from the above discussion. In what follows, we will treat only the difference or variation of this tensor under the gauge as we mentioned above. In that case, the divergence will not appear. By using Eq.(44) and Eq.(46), we have

$$\frac{d}{d\lambda}(\widetilde{h}_\lambda)_{ab}|_{\bar{\lambda}} = \left[\frac{\partial(\widetilde{h}_\lambda)_{ab}}{\partial\rho_\lambda}\frac{d\rho_\lambda}{d\lambda}\right]_{\bar{\lambda}} + \frac{2}{n-1}\rho_{\bar{\lambda}}^{n-1}\mathcal{L}_\zeta(\widetilde{E}_{\bar{\lambda}})_{ab} - \frac{2}{n-1}\rho_{\bar{\lambda}}^{n-1}\frac{d}{d\lambda}(\widetilde{E}_\lambda)_{ab}|_\lambda + O(\rho_{\bar{\lambda}}^n). \tag{48}$$

At this stage, we deduce an important result in this section

$$\frac{d}{d\lambda}(\widetilde{g}_\lambda)_{ab}|_{\bar{\lambda}} = \mathcal{L}_\zeta(\widetilde{g}_{\bar{\lambda}})_{ab} + \frac{2}{n-1}\rho_{\bar{\lambda}}^{n-1}\mathcal{L}_\zeta(\widetilde{E}_{\bar{\lambda}})_{ab} - \frac{2}{n-1}\rho_{\bar{\lambda}}^{n-1}\frac{d}{d\lambda}(\widetilde{E}_\lambda)_{ab}|_{\bar{\lambda}} + O(\rho_{\bar{\lambda}}^n). \tag{49}$$

Thus the variation of $\widetilde{g}_\lambda$ can be splitted into two parts: the first part is given by the dependence of ρ_λ on λ, and can be regarded as a result of trivial differomorphism which is generated by the vector field ζ, and the second part is given by the "dynamical information" of the system. Therefore we will mainly concentrate on the second part below.

3.2 Derivation of Conserved Charges

The parts related to the Lie-derivative in Eq.(49) correspond to the gauge freedoms, they can be gauge fixed away. We fix ρ_λ to be $\rho = \rho_{\bar{\lambda}}$ such that $\zeta = 0$, and consider the variation of the form

$$\frac{d}{d\lambda}(\widetilde{g}_\lambda)_{ab}|_{\bar{\lambda}} = -\frac{2}{n-1}\rho^{n-1}\frac{d}{d\lambda}(\widetilde{E}_\lambda)_{ab}|_{\bar{\lambda}} + O(\rho^n). \tag{50}$$

Recalling that ρ has been regarded as a fixed function which does not depend on λ, after integrating λ from 0 to $\bar{\lambda}$, we have

$$(\widetilde{g}_{\bar{\lambda}})_{ab} - (\widetilde{g}_0)_{ab} = -\frac{2}{n-1}\rho^{n-1}\left[(\widetilde{E}_{\bar{\lambda}})_{ab} - (\widetilde{E}_0)_{ab}\right] + O(\rho^n), \tag{51}$$

where $\widetilde{g}_0$ has the same conformal infinity as $\widetilde{g}_{\bar{\lambda}}$ and plays the role of the metric of the reference background solution. Here we have already assumed that $\widetilde{g}_0$ and $\widetilde{g}_{\bar{\lambda}}$ belong to the same connected component of the solution subspace; therefore, there is a smooth path that connects them. We emphasize that, although $(\widetilde{E}_\lambda)_{ab}$ may be divergent when ρ approaches to zero, the difference $\left[(\widetilde{E}_{\bar{\lambda}})_{ab} - (\widetilde{E}_0)_{ab}\right]$ is always finite once one fixes the conformal factor to be ρ. In other words, the leading order of the variation of $(\widetilde{E}_\lambda)_{ab}$ and $\left[(\widetilde{E}_{\bar{\lambda}})_{ab} - (\widetilde{E}_0)_{ab}\right]$ has the form ρ^{3-n}.

Following [12], we view $\widetilde{g}_{\bar{\lambda}}$ with the expression (51) as a "gauge condition" on the metric, i.e. as a particular representative in the equivalence class of metrics which is diffeomorphic to $g_{\bar{\lambda}}$. The (on-shell) metric variations respecting this gauge choice (with ρ_λ be fixed as ρ) therefore take the form

$$\frac{d}{d\lambda}(g_\lambda)_{ab}|_{\bar{\lambda}} = (\gamma_{\bar{\lambda}})_{ab} + \mathcal{L}_\eta (g_{\bar{\lambda}})_{ab}, \tag{52}$$

where the first piece $(\gamma_{\bar{\lambda}})_{ab}$ is a metric variation of the form

$$(\gamma_{\bar{\lambda}})_{ab} = -\frac{2}{n-1}\rho^{n-1}\frac{d}{d\lambda}(\widetilde{E}_\lambda)_{ab}|_{\bar{\lambda}} + O(\rho^n), \tag{53}$$

and the second piece is an infinitesimal diffeomorphism generated by an arbitrary vector field η respecting the gauge choice, i.e., a diffeo satisfying $\mathcal{L}_\eta g_0 \sim O(\rho^n)$, where g_0 is the metric of reference space-time. Thus,

$$\mathcal{L}_\eta (g_{\bar{\lambda}})_{ab} = -\frac{2}{n-1}\rho^{n-1}\mathcal{L}_\eta\left[(\widetilde{E}_{\bar{\lambda}})_{ab} - (\widetilde{E}_0)_{ab}\right] + O(\rho^n). \tag{54}$$

Inserting these expressions into the definition of the symplectic current form $\boldsymbol{\omega}(g, \delta_1 g, \delta_2 g)$, we see that $\boldsymbol{\omega}|_{\mathcal{B}} = 0$. Hence, the conserved charges H_ξ exist and are indeed conserved.

The variation of Noether charge $(n-2)$-form is [12]

$$\frac{d}{d\lambda}(\mathbf{Q}_\lambda)_{a_1\ldots a_{n-2}}|_{\bar{\lambda}} = \frac{1}{8\pi G}(\widetilde{\boldsymbol{\epsilon}}_{\bar{\lambda}})_{a_1\ldots a_{n-2}bc}(\widetilde{n}_{\bar{\lambda}})^b \frac{d}{d\lambda}(\widetilde{E}_\lambda)^c{}_d \xi^d|_{\bar{\lambda}} + O(\rho)\,. \tag{55}$$

Using the relation

$${}^{(n)}\widetilde{\boldsymbol{\epsilon}}_\lambda = \widetilde{n}_\lambda \wedge {}^{(n-1)}\widetilde{\boldsymbol{\epsilon}} = \widetilde{n}_\lambda \wedge \widetilde{u} \wedge {}^{(n-2)}\widetilde{\boldsymbol{\epsilon}}, \tag{56}$$

among the n-dimensional volume form, the induced $(n-1)$-dimensional volume form of the boundary $\mathcal{B}$ and the $(n-2)$-dimensional volume form of the cross section $\partial\Sigma$, we can rewrite Eq.(55) as

$$\frac{d}{d\lambda}(\mathbf{Q}_\lambda)_{a_1\ldots a_{n-2}}|_{\bar{\lambda}} = \frac{-1}{8\pi G}\frac{d}{d\lambda}\left[{}^{(n-2)}\widetilde{\boldsymbol{\epsilon}}_{a_1\ldots a_{n-2}}\,(\widetilde{E}_\lambda)^c{}_d \widetilde{u}_c \xi^d\right]_{\bar{\lambda}}\,, \quad \text{on } \mathcal{B}. \tag{57}$$

A similar calculation can be done and gives $\boldsymbol{\Theta}_{\bar{\lambda}}|_{\mathcal{B}} = 0$. Thus, we have

$$\frac{d}{d\lambda}H_\xi[g_\lambda]_{\bar{\lambda}} = \int_{\partial\Sigma}\frac{d}{d\lambda}\mathbf{Q}_\lambda|_{\bar{\lambda}} = \frac{-1}{8\pi G}\,\left[\frac{d}{d\lambda}\int_{\partial\Sigma}(\widetilde{E}_\lambda)_{ab}\widetilde{u}^b\xi^a\, d\widetilde{S}\right]_{\bar{\lambda}}. \tag{58}$$

Integrating this equation over λ from $\lambda = 0$ to $\bar{\lambda}$, yields

$$H_\xi[g_{\bar{\lambda}}] - H_\xi[g_0] = \frac{-1}{8\pi G}\,\int_{\partial\Sigma}\left[(\widetilde{E}_{\bar{\lambda}})_{ab} - (\widetilde{E}_0)_{ab}\right]\widetilde{u}^b\xi^a\, d\widetilde{S}. \tag{59}$$

Choosing $H_\xi[g_0] = 0$ for all asymptotic symmetric representatives ξ^a, we get the result

$$H_\xi[g_{\bar{\lambda}}] \;=\; \frac{-1}{8\pi G}\,\int_{\partial\Sigma}\left[(\widetilde{E}_{\bar{\lambda}})_{ab} - (\widetilde{E}_0)_{ab}\right]\widetilde{u}^b\xi^a\, d\widetilde{S}, \tag{60}$$

which can also be expressed by Weyl tensor

$$H_\xi[g_{\bar\lambda}] = \frac{-1}{8\pi G(n-3)} \int_{\partial\Sigma} \lim_{\to\mathcal{B}} \rho^{3-n} \left[(\widetilde{C}_{\bar\lambda})_{acbd}(\widetilde{n}_{\bar\lambda})^c(\widetilde{n}_{\bar\lambda})^d - (\widetilde{C}_0)_{acbd}(\widetilde{n}_0)^c(\widetilde{n}_0)^d\right] \widetilde{u}^b\xi^a\, d\widetilde{S}. \tag{61}$$

This is nothing, but the one we obtained in this paper for the expression of conserved charges for AlAdS space-times, This expression is conformal invariant, so in fact we can choose a more simple conformal factor instead of ρ (for example, $\Omega = 1/r$ in Sec. 2) to calculate the conserved charges. In AAdS space-times, the reference space-time is fixed to be an exact AdS space-time which is conformal flat, and in this case the above formula (61) reduces to the definition of Ashtekar *et al* [10]. For a general AlAdS space-time, however, one has no prior background which can be chosen as an appropriate reference background solution.

Although we have discussed only the even dimensional cases, it is easy to see that the above procedure can be extended to the more general AlAdS cases where the Fefferman-Graham expansion like (32) without log terms can be implemented. For example, one can find that in the static AlAdS space-time setting with Ricci flat boundary, for both even and odd dimensional cases, no log terms arise in the expansion [26]. Thus, the same conserved charges for those space-times can be defined as Eq.(61). As an example, in Sec. 7 we will calculate conserved charges for (un)warped black brane space-times, which belong to such a kind of space-times.

4 Taub-Nut-AdS and Taub-Bolt-AdS Space-times

Taub-Nut-AdS and Taub-Bolt-AdS solutions are AlAdS solutions. For example, in the Euclidean sector, their boundaries are $U(1)$ bundles over $\underbrace{S^2 \times \cdots \times S^2}_{k}$ for $2k+2$ dimension[3], their metrics can be expressed as [27]

$$ds^2 = F(r)(d\tau + 2n\cos\theta_i d\phi_i)^2 + \frac{dr^2}{F(r)} + (r^2 - n^2)\sum_{i=1}^{i=k}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2), \tag{62}$$

where i is summed from 1 to k and $F(r)$ is given by

$$F(r) = \frac{r}{(r^2-n^2)^k}\int^r \left[\frac{(s^2-n^2)^k}{s^2} + \frac{2k+1}{\ell^2}\frac{(s^2-n^2)^{k+1}}{s^2}\right] ds - \frac{2Mr}{(r^2-n^2)^k}. \tag{63}$$

One can find $F(r) \sim \frac{r^2}{\ell^2}$ when r approaches to infinity. If we choose the conformal factor as $\Omega = \frac{\ell}{r}$, the boundary metrics have the form

$$d\bar{s}^2|_{\mathcal{B}} = \Omega^2 ds^2|_{\mathcal{B}} \sim (d\tau + 2n\cos\theta_i d\phi_i)^2 + \ell^2\left(d\theta_i{}^2 + \sin^2\theta_i d\phi_i{}^2\right). \tag{64}$$

[3]Where $S^2 \times \cdots \times S^2$ is called base space. It may be other Einstein-Kähler manifold, for example $T^2 \times \cdots \times T^2$, $S^2 \times \cdots \times S^2 \times T^2 \times \cdots \times T^2$ and $CP(k)$. We only consider the sphere cases in this paper.

It is easy to find that these boundaries are trivial bundles (i.e., direct product manifolds $S^1 \times S^2 \times \cdots \times S^2$) when the Nut charges vanish. Therefore non-vanishing Nut charges give rise to the non-triviality of the bundles.

Before calculating their conserved charges, the following points are worthwhile to stress: (i) Note that the Nut charges n appear in the boundary metrics, we conclude that solutions with different nut charges have different boundary metrics. Thus we can not view a solution with Nut charge n_1 as a reference solution for calculating conserved charges of another solution with different Nut charge n_2. This is very different from the AAdS case where the boundary is fixed to be Einstein static space-time. (ii) Since our formula is background dependent, there is some freedom to choose the reference solution. Hawking *et al.* [28] and Chamblin *et al.* [29] have argued that one can use 4-dimensional Taub-Nut-AdS solution as the reference solution, when considering Taub-Bolt-AdS black hole solutions. They have calculated the corresponding thermodynamic quantities of 4-dimensional Taub-Bolt-AdS solution by using "background subtraction" method. The resultant thermodynamic quantities obey the first law of black hole thermodynamics. Using "Noether charge" method Clarkson *et al.* [30] have computed the conserved charges (Q_{Noether}) by treating the Nut solutions as reference solutions in higher dimensional cases. They also have computed these quantities ($M_{\mathrm{bolt}}, M_{\mathrm{Nut}}$) by using the "boundary counterterm" method and found a relation between them:

$$Q_{\mathrm{Noether}} = M_{\mathrm{bolt}} - M_{\mathrm{Nut}} \tag{65}$$

Motivated by these works and to compare our method with other methods, in this section, we will select the Nut solutions as reference solutions. Then we calculate the conserved quantities for $4-, 6-, 8-$ and 10-dimensional cases by using our new formula (61). Our results agree with those given in [30] in any dimension. Furthermore, let us note that if chooses a massless solution, but with the same Nut charge, as the reference solution, one can also get finite results, as shown below. In that case, our results are the same as those resulting from the "boundary counterterm" method. For simplicity, we will mainly consider the Euclidean sector for these metrics. The quantities in the Lorentzian sector can be obtained by analytically continuing the coordinate τ and also the parameter n (i.e., one replaces n^2 with $-N^2$).

4.1 Four Dimensional Solutions

The four dimensional Nut charged AdS solution has the following form [27]

$$ds^2 = F(r)(d\tau + 2n\cos\theta d\phi)^2 + F(r)^{-1}dr^2 + (r^2 - n^2)(d\theta^2 + \sin^2\theta d\phi^2), \tag{66}$$

where $F(r)$ is given by

$$F(r) = \frac{1}{\ell^2(r^2-n^2)}\left[\ell^2(r^2+n^2) - 2Mr\ell^2 + (r^4 - 6n^2r^2 - 3n^4)\right]. \tag{67}$$

In order for this solution to describe a Nut solution, the mass parameter has to be fixed to be

$$M_n = \frac{n(\ell^2-4n^2)}{\ell^2}, \tag{68}$$

so that $F(r=n)=0$ and the dimension of the fixed-point set of ∂_τ is zero. Fixing the mass parameter at this value, $F(r)$ is then given by

$$F_n(r) = \frac{r-n}{r+n}(1+\ell^{-2}(r-n)(r+3n)). \tag{69}$$

On the other hand, the Bolt solution is given by

$$F(r) = F_b(r) = \frac{r^2 - 2M_b r + n^2 + \ell^{-2}(r^4 - 6n^2r^2 - 3n^4)}{r^2-n^2}, \tag{70}$$

where

$$M_b = \frac{r_b^2+n^2}{2r_b} + \frac{1}{2\ell^2}(r_b^3 - 6n^2 r_b - 3\frac{n^4}{r_b}), \tag{71}$$

with

$$r_{b\pm} = \frac{\ell^2}{12n}(1 \pm \sqrt{1 - 48\frac{n^2}{\ell^2} + 144\frac{n^4}{\ell^4}}). \tag{72}$$

For r_b to be real the discriminant must be no-negative. Furthermore, the condition must be satisfied: $r_b > n$, which gives

$$n \leq (\frac{1}{6} - \frac{\sqrt{3}}{12})^{1/2}\ell. \tag{73}$$

Treating the Nut solution as the reference solution, we give the conserved charges of the Taub-Bolt-AdS solution, corresponding to the Killing vector ∂_τ, as [4]

$$H_{\partial_\tau}[g_1] = \frac{\ell}{8\pi G}\int_{\partial\Sigma}\Omega^{-1}\left[(\widetilde{C}_1)_{acbd}(\tilde{n}_1)^c(\tilde{n}_1)^d - (\widetilde{C}_0)_{acbd}(\tilde{n}_0)^c(\tilde{n}_0)^d\right]\tilde{u}^b(\partial_\tau)^a\, d\widetilde{S}, \tag{74}$$

where the subscripts "1" and "0" correspond to "Bolt" and "Nut" solution, respectively. By straightforward calculations, it turns out that the leading order of the Weyl tensor for the solution (66) is

$$\widetilde{C}^\tau{}_{r\tau r} = C^\tau{}_{r\tau r} \sim \frac{2M\ell^2}{r^5} + O(\frac{1}{r^6}). \tag{75}$$

Choosing the conformal factor as $\frac{\ell}{r}$, one then has

$$\begin{aligned}(\widetilde{E}_1)^\tau{}_\tau - (\widetilde{E}_0)^\tau{}_\tau &= \Omega^{-1}\left[(C_1)^\tau{}_{r\tau r}(\tilde{n}_1)^r(\tilde{n}_1)^r - (C_0)^\tau{}_{r\tau r}(\tilde{n}_0)^r(\tilde{n}_0)^r\right] \\ &\sim \frac{r^5}{\ell^5}\frac{2\ell^2(M_b-M_n)}{r^5} \\ &= \frac{2(M_b-M_n)}{\ell^3}.\end{aligned} \tag{76}$$

[4]Hereafter we use the orientation convention of Gibbons *et al.* [31] such that $d\widetilde{S}_t$ is positive.

Note that $d\widetilde{S}_\tau = \ell^2 \sin\theta d\theta$, we obtain the mass of the Taub-Bolt-AdS solution

$$\begin{aligned} \Delta M &= \frac{\ell}{8\pi G}\int_{\partial\Sigma} \frac{2(M_b - M_n)}{\ell^3}\ell^2 \sin\theta d\theta d\phi \\ &= \frac{(M_b - M_n)}{G}. \end{aligned} \tag{77}$$

This is completely in agreement with the one in [29, 30]. If one writes the metric in the form in [28], the same result as in [28] can also be obtained.

4.2 Six Dimensional Solutions

The six dimensional Nut charged AdS solution has the metric [27]

$$\begin{aligned} ds^2 &= F(r)(d\tau + 2n\cos\theta_1 d\phi_1 + 2n\cos\theta_2 d\phi_2)^2 + F(r)^{-1}dr^2 \\ &\quad +(r^2 - n^2)(d\theta_1{}^2 + \sin^2\theta_1 d\phi_1{}^2 + d\theta_2{}^2 + \sin^2\theta_2 d\phi_2{}^2), \end{aligned} \tag{78}$$

where $F(r)$ is given by

$$F(r) = \frac{1}{3\ell^2(r^2-n^2)^2}\left[3r^6 + (\ell^2 - 15n^2)r^4 - 3n^2(2\ell^2 - 15n^2)r^2 - 6Mr\ell^2 - 3n^4(\ell^2 - 5n^2)\right]. \tag{79}$$

When the mass parameter M is fixed to be

$$M_n = \frac{4n^3(6n^2 - \ell^2)}{3\ell^2}, \tag{80}$$

this solution describes a Nut solution with $F(r = n) = 0$, so that the dimension of the fixed-point set of ∂_τ is zero. Fixing the mass at this value, $F(r)$ is changed to

$$F_n(r) = \frac{(r-n)(3r^3 + 9nr^2 + (\ell^2 + 3n^2)r + 3n(\ell^2 - 5n^2))}{3(r+n)^2\ell^2}. \tag{81}$$

A regular Bolt solution has the mass parameter

$$M = M_b = \frac{-1}{6\ell^2}[3r_b{}^5 + (\ell^2 - 15n^2)r_b{}^3 - 3n^2(2\ell^2 - 15n^2)r_b - 3n^4(\ell^2 - 5n^2)/r_b], \tag{82}$$

where r_b is a function of n and ℓ

$$r_{b\pm} = \frac{1}{30n}\left(\ell^2 \pm \sqrt{\ell^4 - 180n^2\ell^2 + 900n^4}\right). \tag{83}$$

To have a real value of r_b the discriminant in the above equation must be non-negative. The condition $r_b > n$ leads to

$$n \leq \left(\frac{3 - 2\sqrt{2}}{30}\right)^{\frac{1}{2}} \ell. \tag{84}$$

Treating the Nut solution as the reference solution, we can give the conserved charges for the Bolt solution

$$H_{\partial_\tau}[g_1] = \frac{\ell}{8\pi G \cdot 3} \int_{\partial\Sigma} \Omega^{-3} \left[(\widetilde{C}_1)_{acbd} (\tilde{n}_1)^c (\tilde{n}_1)^d - (\widetilde{C}_0)_{acbd} (\tilde{n}_0)^c (\tilde{n}_0)^d \right] \tilde{u}^b (\partial_\tau)^a \, d\widetilde{S}. \tag{85}$$

By straightforward calculation, the first two terms of the Weyl tensor for the solution (78) are

$$\widetilde{C}^\tau{}_{r\tau r} = C^\tau{}_{r\tau r} \sim \frac{4n^2(\ell^2 - 6n^2)}{r^6} + \frac{12\ell^2 M}{r^7} + O(\frac{1}{r^8}). \tag{86}$$

Note that the first term on the right hand side of the above equation (86) is independent of the mass parameter M. Thus if we directly apply the formula given by Ashtekar *et al.* [10] to this six dimensional Bolt solution, obviously we will get a divergent result due to (86). However, using our formula (61), we have

$$\begin{aligned} (\widetilde{E}_1)^\tau{}_\tau - (\widetilde{E}_0)^\tau{}_\tau &= \frac{1}{3} \Omega^{-3} \left[(C_1)^\tau{}_{r\tau r} (\tilde{n}_1)^r (\tilde{n}_1)^r - (C_0)^\tau{}_{r\tau r} (\tilde{n}_0)^r (\tilde{n}_0)^r \right] \\ &\sim \frac{r^7}{\ell^7} \frac{4\ell^2(M_b - M_n)}{r^7} \\ &= \frac{4(M_b - M_n)}{\ell^5}. \end{aligned} \tag{87}$$

Using $\sqrt{g} = (r^2 - n^2)^2 \sin\theta_1 \sin\theta_2$, one has

$$d\widetilde{S}_\tau = \ell^4 \sin\theta_1 \sin\theta_2 d\theta_1 d\theta_2 d\phi_1 d\phi_2. \tag{88}$$

Finally we obtain the mass of the six dimensional Bolt solution

$$\begin{aligned} \Delta M &= \frac{\ell}{8\pi G} \int_{\partial\Sigma} \frac{4(M_b - M_n)}{\ell^5} \ell^4 \sin\theta_1 \sin\theta_2 d\theta_1 d\theta_2 d\phi_1 d\phi_2 \\ &= \frac{8\pi}{G}(M_b - M_n). \end{aligned} \tag{89}$$

Again, it is identical to the one in [30].

4.3 Eight Dimensional Solutions

The eight dimensional Nut charged AdS solution has the following form [27]

$$\begin{aligned} ds^2 &= F(r)(d\tau + 2n\cos\theta_1 d\phi_1 + 2n\cos\theta_2 d\phi_2 + 2n\cos\theta_3 d\phi_3)^2 + F(r)^{-1} dr^2 \\ &\quad + (r^2 - n^2)(d\theta_1{}^2 + \sin^2\theta_1 d\phi_1{}^2 + d\theta_2{}^2 + \sin^2\theta_2 d\phi_2{}^2 + d\theta_3{}^2 + \sin^3\theta_3 d\phi_3{}^2), \end{aligned} \tag{90}$$

where $F(r)$ is given by

$$F(r) = \frac{5r^8 + (\ell^2 - 28n^2)r^6 + 5n^2(14n^2 - \ell^2)r^4 + 5(3\ell^2 - 28n^2)r^2 - 10Mr\ell^2 + 5n^6(\ell^2 - 7n^2)}{5\ell^2(r^2 - n^2)^3}. \tag{91}$$

In order to have a Nut solution, the mass parameter M must be fixed as

$$M_n = \frac{8n^5(\ell^2 - 8n^2)}{5\ell^2}. \tag{92}$$

Once again by fixing the mass at the above value, the function $F(r)$ is then

$$F(r) = \frac{(r-n)(5r^4 + 20nr^3 + (\ell^2 + 22n^2)r^2 + (4n\ell^2 - 12n^3)r - 35n^4 + 5\ell^2 n^2)}{5(r+n)^3\ell^2}. \tag{93}$$

In order to have a regular Bolt solution we must impose the mass M as

$$M_b = \frac{1}{10\ell^2}[{r_b}^7 + (\ell^2 - 28n^2){r_b}^5 + 5n^2(14n^2 - \ell^2){r_b}^3 + 5(3\ell^2 - 28n^2)r_b + 5n^6(\ell^2 - 7n^2)/r_b], \tag{94}$$

where r_b is

$$r_{b\pm} = \frac{1}{56n}\left(\ell^2 \pm \sqrt{\ell^4 - 448n^2\ell^2 + 3136n^4}\right). \tag{95}$$

Requiring that r_b be real and also be greater than n implies that

$$n \leq \left(\frac{4 - \sqrt{15}}{56}\right)^{\frac{1}{2}} \ell. \tag{96}$$

Treating the Nut solution as a reference solution, we can get the conserved charges for the Bolt solution as

$$H_{\partial_\tau}[g_1] = \frac{\ell}{8\pi G \cdot 5}\int_{\partial\Sigma} \Omega^{-5}\left[(\widetilde{C}_1)_{acbd}(\widetilde{n}_1)^c(\widetilde{n}_1)^d - (\widetilde{C}_0)_{acbd}(\widetilde{n}_0)^c(\widetilde{n}_0)^d\right]\widetilde{u}^b(\partial_\tau)^a\, d\widetilde{S}. \tag{97}$$

Note that in this case one has the following component of the Weyl tensor for the solution (90)

$$\begin{aligned}
\widetilde{C}^\tau{}_{r\tau r} = C^\tau{}_{r\tau r} \quad \sim \quad & \frac{6/5n^2(\ell^2 - 8n^2)}{r^6} \\
& -\frac{3/175(14\ell^4 n^2 + \ell^2(1875 - 1469n^4) + 28n^2(-625 + 477n^4))}{r^8} \\
& +\frac{30M\ell^2}{r^9} + O(\frac{1}{r^{10}}),
\end{aligned} \tag{98}$$

and then

$$\begin{aligned}
(\widetilde{E}_1)^\tau{}_\tau - (\widetilde{E}_0)^\tau{}_\tau &= \frac{1}{5}\Omega^{-5}\left[(C_1)^\tau{}_{r\tau r}(\widetilde{n}_1)^r(\widetilde{n}_1)^r - (C_0)^\tau{}_{r\tau r}(\widetilde{n}_0)^r(\widetilde{n}_0)^r\right] \\
&\sim \frac{r^9}{\ell^9}\frac{6\ell^2(M_b - M_n)}{r^9} \\
&= \frac{6(M_b - M_n)}{\ell^7}.
\end{aligned} \tag{99}$$

With $\sqrt{g} = (r^2 - n^2)^3 \sin\theta_1 \sin\theta_2 \sin\theta_3$, and

$$d\widetilde{S}_\tau = \ell^6 \sin\theta_1 \sin\theta_2 \sin\theta_3 d\theta_1 d\theta_2 d\theta_3 d\phi_1 d\phi_2 d\phi_3, \tag{100}$$

we obtain the energy of the eight dimensional Bolt solution

$$\begin{aligned}
\Delta M &= \frac{\ell}{8\pi G}\int_{\partial\Sigma}\frac{6(M_b-M_n)}{\ell^7}\ell^6\sin\theta_1\sin\theta_2\sin\theta_3 d\theta_1 d\theta_2 d\theta_3 d\phi_1 d\phi_2 d\phi_3 \\
&= \frac{48\pi^2}{G}(M_b-M_n). \qquad (101)
\end{aligned}$$

4.4 Ten Dimensional Solutions

Ten dimensional Nut charged AdS solution is given by [27]

$$\begin{aligned}
ds^2 &= F(r)(d\tau+2n\cos\theta_1 d\phi_1+2n\cos\theta_2 d\phi_2+2n\cos\theta_3 d\phi_3+2n\cos\theta_4 d\phi_4)^2 \\
&+ F(r)^{-1}dr^2+(r^2-n^2)(d\theta_1{}^2+\sin^2\theta_1 d\phi_1{}^2+d\theta_2{}^2+\sin^2\theta_2 d\phi_2{}^2 \\
&+ d\theta_3{}^2+\sin^2\theta_3 d\phi_3{}^2+d\theta_4{}^2+\sin^2\theta_4 d\phi_4{}^2), \qquad (102)
\end{aligned}$$

where $F(r)$ has the form

$$\begin{aligned}
F(r) &= \frac{1}{35\ell^2(r^2-n^2)^4}[35r^{10}+5(\ell^2-45n^2)r^8+14n^2(45n^2-2\ell^2)r^6 \\
&+ 70n^4(\ell^2-15n^2)r^4+35n^6(45n^2-4\ell^2)r^2-70Mr\ell^2+35n^8(9n^2-\ell^2)]. \qquad (103)
\end{aligned}$$

In order to describe a Nut solution, the mass parameter M must be fixed as

$$M_n=\frac{64n^7(10n^2-\ell^2)}{35\ell^2}. \qquad (104)$$

Taking the mass parameter to be the above value, one has $F(r)$ as

$$\begin{aligned}
F(r) &= \frac{1}{35(r+n)^4\ell^2}(r-n)(35r^5+175nr^4+(300n^2+5\ell^2)r^3 \\
&+ (25n\ell^2+100n^3)r^2+(47n^2\ell^2-295n^4)r-315n^5+35\ell^2n^3). \qquad (105)
\end{aligned}$$

In order to have a regular Bolt solution we must impose the mass M as

$$\begin{aligned}
M_b &= \frac{1}{70\ell^2}[35r_b{}^9+(5\ell^2-225n^2)r_b{}^7+n^2(630n^2-28\ell^2)r_b{}^5+n^4(70\ell^2-1050n^2)r_b{}^3 \\
&+ n^6(1575n^2-140\ell^2)r_b+n^8(315n^2-35\ell^2)], \qquad (106)
\end{aligned}$$

where

$$r_{b\pm}=\frac{1}{90n}\left(\ell^2\pm\sqrt{\ell^4-900^2\ell^2+8100n^4}\right). \qquad (107)$$

Requiring that r_b is real and larger than n implies

$$n\leq\left(\frac{5-2\sqrt{6}}{90}\right)^{\frac{1}{2}}\ell. \qquad (108)$$

In this case we have the conserved charge of the Bolt solution by considering the Nut solution as a reference solution

$$H_{\partial_\tau}[g_1] = \frac{\ell}{8\pi G \cdot 7} \int_{\partial\Sigma} \Omega^{-7} \left[(\widetilde{C}_1)_{acbd} (\widetilde{n}_1)^c (\widetilde{n}_1)^d - (\widetilde{C}_0)_{acbd} (\widetilde{n}_0)^c (\widetilde{n}_0)^d \right] \widetilde{u}^b (\partial_\tau)^a \, d\widetilde{S}. \tag{109}$$

Note that here we have

$$\begin{aligned}
\widetilde{C}^{\tau}{}_{r\tau r} = C^{\tau}{}_{r\tau r} \quad \sim & \quad \frac{24n^2\left(\ell^2 - 10n^2\right)}{35r^6} \\
+ & \quad \frac{8\left(-3\ell^4 n^2 + 11\ell^2 n^4 + 190n^6\right)}{245r^8} \\
+ & \quad \frac{8\left(15\ell^6 n^2 - 142\ell^4 n^4 + 22350\ell^2 n^6 - 224300n^8\right)}{8575r^{10}} \\
+ & \quad \frac{56\ell^2 M}{r^{11}} + O(\frac{1}{r^{12}}),
\end{aligned} \tag{110}$$

and

$$\begin{aligned}
(\widetilde{E}_1)^{\tau}{}_{\tau} - (\widetilde{E}_0)^{\tau}{}_{\tau} &= \frac{1}{7}\Omega^{-7}\left[(C_1)^{\tau}{}_{r\tau r}(\widetilde{n}_1)^r(\widetilde{n}_1)^r - (C_0)^{\tau}{}_{r\tau r}(\widetilde{n}_0)^r(\widetilde{n}_0)^r\right] \\
&\sim \frac{r^9}{\ell^9}\frac{8\ell^2(M_b - M_n)}{r^9} \\
&= \frac{8(M_b - M_n)}{\ell^7}.
\end{aligned} \tag{111}$$

In ten dimensional case, one has $\sqrt{g} = (r^2 - n^2)^4 \sin\theta_1 \sin\theta_2 \sin\theta_3 \sin\theta_4$, and

$$d\widetilde{S}_\tau = \ell^8 \sin\theta_1 \sin\theta_2 \sin\theta_3 \sin\theta_4 d\theta_1 d\theta_2 d\theta_3 d\theta_4 d\phi_1 d\phi_2 d\phi_3 d\phi_4. \tag{112}$$

Thus, we obtain the mass of the ten dimensional Bolt solution

$$\begin{aligned}
\Delta M &= \frac{\ell}{8\pi G}\int_{\partial\Sigma} \frac{8(M_b - M_n)}{\ell^9} \ell^8 \sin\theta_1 \sin\theta_2 \sin\theta_3 d\theta_1 d\theta_2 d\theta_3 d\phi_1 d\phi_2 d\phi_3 \\
&= \frac{256\pi^3}{G}(M_b - M_n).
\end{aligned} \tag{113}$$

At the end of this section, we give the mass formula for a general Nut charged AdS solution in $2k+2$ dimension, which has the form (62). The Nut solution is obtained by fixing the mass parameter as [30]

$$M_n = \frac{n^{2k-1}}{\sqrt{\pi}\ell^2}\left[\ell^2 - (2k+2)n^2\right] \frac{\Gamma\left(\frac{3-2k}{2}\right)\Gamma(k+1)}{(2k-1)}, \tag{114}$$

while the Bolt solution corresponds to the case with the mass parameter

$$M_b = \frac{1}{2}\left[\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^i n^{2i} r_b^{2k-2i-1}}{(2k-2i-1)} + \frac{(2k+1)}{\ell^2}\sum_{i=0}^{k+1}\binom{k+1}{i}\frac{(-1)^i n^{2i} r_b^{2k-2i+1}}{(2k-2i+1)}\right], \tag{115}$$

where $r_b > n$ is determined by $F(r_b) = 0$ and $F'(r_b) = \frac{2}{n(2k+2)}$. For these general Taub-Bolt-AdS and Taub-Nut-AdS solutions we find

$$\widetilde{E}^{\tau}{}_{1,\tau} - \widetilde{E}^{\tau}{}_{0,\tau} = \frac{(n-2)(M_b - M_n)}{\ell^{n+1}}, \qquad \Delta M = \frac{(n-2)(4\pi)^{(n-2)/2}(M_b - M_n)}{8\pi G}. \tag{116}$$

This result coincides with that in [30], where those authors get the result by using "boundary counterterm" method and "Noether method".

5 Four Dimensional Kerr-Taub-Nut-AdS Solution

In this section we discuss the case of four dimensional Euclidean Kerr-Taub-Nut-AdS space-times, which has the form [32]

$$\begin{aligned} ds^2 &= \frac{V(r)\,(d\tau - (2N\cos\theta - a\sin^2\theta)d\phi)^2) + \mathcal{H}(\theta)\sin^2\theta\,(a\,d\tau - (r^2 - N^2 - a^2)\,d\phi)^2}{\chi^4\,(r^2 - (N + a\,\cos\theta)^2)} \\ &\quad + (r^2 - (N + a\,\cos\theta)^2)\left(\frac{dr^2}{V(r)} + \frac{d\theta^2}{\mathcal{H}(\theta)}\right), \end{aligned} \tag{117}$$

where

$$\begin{aligned} \mathcal{H}(\theta) &= 1 + \frac{q\,N^2}{\ell^2} + \frac{(2\,N + a\,\cos\theta)^2}{\ell^2}, \\ V(r) &= \frac{r^4}{\ell^2} + \frac{((q-2)N^2 - a^2 + \ell^2)\,r^2}{\ell^2} - 2\,M\,r - \frac{(a+N)\,(a-N)\,(q\,N^2 + \ell^2 + N^2)}{\ell^2}. \end{aligned} \tag{118}$$

The periodicity in τ and the parameters q and χ are chosen so that conical singularities are avoided. In the (θ, ϕ) section these considerations imply that $q = -4$ and $\chi = 1/\sqrt{1 + a^2/l^2}$.

For this solution, a straightforward calculation gives

$$\begin{aligned} \widetilde{C}^{\tau}{}_{r\tau r} &= C^{\tau}{}_{r\tau r} \sim \frac{2M\ell^2}{r^5} + O(\frac{1}{r^6}), \\ \widetilde{C}^{\tau}{}_{r\phi r} &= C^{\tau}{}_{r\phi r} \sim \frac{3M\ell^2(a\sin^2\theta - 2N\cos\theta)}{r^5} + O(\frac{1}{r^6}). \end{aligned} \tag{119}$$

The conformal boundary volume has the form

$$d\widetilde{S}_{\tau} = \frac{\ell^2}{\chi^4}\sin\theta d\theta d\phi. \tag{120}$$

In this case we choose the massless solution, namely the case with $M = 0$, as a reference solution. Thus we find

$$\begin{aligned} (\widetilde{E}_1)^{\tau}{}_{\tau} - (\widetilde{E}_0)^{\tau}{}_{\tau} &= \Omega^{-1}\left[(C_1)^{\tau}{}_{r\tau r}(\tilde{n}_1)^r(\tilde{n}_1)^r - (C_0)^{\tau}{}_{r\tau r}(\tilde{n}_0)^r(\tilde{n}_0)^r\right] \\ &\sim \frac{r^5}{\ell^5}\frac{2\ell^2 M}{r^5} = \frac{2M}{\ell^3}, \end{aligned} \tag{121}$$

and the conserved charge associated to ∂_τ

$$H_{\partial_\tau} = \frac{\ell}{8\pi G}\int_{\partial\Sigma} \frac{2M}{\ell^3}\frac{\ell^2}{\chi^4}\sin\theta d\theta d\phi = \frac{1}{G}\frac{M}{\chi^4}. \tag{122}$$

Similarly we have

$$\begin{aligned}(\widetilde{E}_1)^\tau{}_\phi - (\widetilde{E}_0)^\tau{}_\phi &= \Omega^{-1}\left[(C_1)^\tau{}_{r\phi r}(\widetilde{n}_1)^r(\widetilde{n}_1)^r - (C_0)^\tau{}_{r\phi r}(\widetilde{n}_0)^r(\widetilde{n}_0)^r\right] \\ &\sim \frac{r^5}{\ell^5}\frac{3M(a\sin^2\theta - 2N\cos^2\theta)\ell^2}{r^5} = \frac{3M(a\sin^2\theta - 2N\cos^2\theta)}{\ell^3},\end{aligned} \tag{123}$$

$$H_{\partial_\phi} = \frac{\ell}{8\pi G}\int_{\partial\Sigma} \frac{3M(a\sin^2\theta - 2N\cos^2\theta)}{\ell^3}\frac{\ell^2}{\chi^4}\sin\theta d\theta d\phi = \frac{1}{G}\frac{Ma}{\chi^4}. \tag{124}$$

These results are just what Mann obtained in [32], but he used the "boundary counterterm" method.

It should be noted here, that the energy E and angular momentum J_ϕ are defined as $E = H_{\partial_t}$ and $J_\phi = -H_{\partial_\phi}$ in the Lorentz sector. The relative sign difference between definitions of energy and angular momentum can be traced back to its origin for the Lorentz signature of the space-time metric as mentioned in [19]. This can be understood by noting the definitions of energy and angular momentum of a particle in special relativity: $E = -p_a t^a$ and $J = +p_a\phi^a$. In the Euclidean sector, the relative sign difference disappears, and the Hamilton associated to ∂_ϕ is just the angular momentum. In the next section, we will calculate the energy and angular momentum of higher dimensional Kerr-AdS solutions with Nut charges, and the relative sign difference will appear because we consider the solutions in the Lorentz sector.

6 Higher Dimensional Kerr-AdS Solutions with Nut Charges

The higher dimensional Kerr-AdS solution with Nut charges has been given recently in [33]

$$ds^2 = \frac{p^2+q^2}{X}dp^2 + \frac{p^2+q^2}{Y}dq^2 + \frac{X}{p^2+q^2}(d\tau + q^2 d\sigma)^2 - \frac{Y}{p^2+q^2}(d\tau - p^2 d\sigma)^2 + \frac{p^2q^2}{\gamma}d\Omega_k^2, \tag{125}$$

where

$$X = \gamma - \epsilon p^2 + \frac{1}{\ell^2}p^4 + 2Np^{1-k}\,, \qquad Y = \gamma + \epsilon q^2 + \frac{1}{\ell^2}q^4 - 2m\,q^{1-k}, \tag{126}$$

and $d\Omega_k^2$ is the metric on the unit sphere S^k, (γ, m, N) are three independent continuous parameters, which are related to the angular momentum, mass and Nut charge, respectively, ϵ is a dimensionless constant, and ℓ is the AdS radius. If we take the parameters in (126) to be

$$\gamma = a^2\,, \qquad \epsilon = 1 + \frac{1}{\ell^2}a^2\,, \qquad m = M, \tag{127}$$

define

$$\begin{aligned}
\Delta_r &= (r^2+a^2)(1+\frac{r^2}{\ell^2}) - 2Mr^{1-k}, \\
\Delta_\theta &= 1-\frac{a^2}{\ell^2}\cos^2\theta, \\
\Xi &= 1-\frac{a^2}{\ell^2}, \\
\rho^2 &= r^2+a^2\cos^2\theta,
\end{aligned} \tag{128}$$

and choose a set of new coordinates as

$$p = a\,\cos\theta\,, \qquad q=r\,, \qquad \tau = t-\frac{a}{\Xi}\,\phi\,, \qquad \sigma = -\frac{1}{a\,\Xi}\,\phi\,, \tag{129}$$

then we can rewrite the solution (125) in the form

$$\begin{aligned}
ds^2 &= -\frac{\Delta_r}{\rho^2}\left(dt-\frac{a}{\Xi}\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta + \frac{2N(a\cos\theta)^{1-k}}{a^2\sin^2\theta}}d\theta^2 \\
&+ \frac{\Delta_\theta\sin^2\theta + \frac{2N(a\cos\theta)^{1-k}}{a^2}}{\rho^2}\left(adt-\frac{a^2+r^2}{\Xi}d\phi\right)^2 + r^2\cos^2\theta d\Omega_k^2.
\end{aligned} \tag{130}$$

When $N=0$, this metric reduces to the metric obtained in [34], which describes a higher dimensional Kerr-AdS black hole with a single rotation parameter. Note that in the four dimensional case, this solution (130) seems not completely the same as that given in (117).

Taking the massless Kerr-Taub-Nut-AdS as the reference solution, namely the solution with $M=0$, we can calculate the conserved quantities associated to killing vector fields ∂_t and ∂_ϕ of the solution (125) in the coordinates $(t,r,\theta,\phi,\cdots)$, according to the following formulas

$$H_{\partial_t}[g_1] = \frac{\ell}{8\pi G(n-3)}\int_{\partial\Sigma}\lim_{\to\mathcal{B}}\Omega^{3-n}\left[(\widetilde{C}_1)_{acbd}(\tilde{n}_1)^c(\tilde{n}_1)^d - (\widetilde{C}_0)_{acbd}(\tilde{n}_0)^c(\tilde{n}_0)^d\right]\tilde{u}^b(\partial_t)^a\,d\widetilde{S}, \tag{131}$$

and

$$H_{\partial_\phi}[g_1] = \frac{\ell}{8\pi G(n-3)}\int_{\partial\Sigma}\lim_{\to\mathcal{B}}\Omega^{3-n}\left[(\widetilde{C}_1)_{acbd}(\tilde{n}_1)^c(\tilde{n}_1)^d - (\widetilde{C}_0)_{acbd}(\tilde{n}_0)^c(\tilde{n}_0)^d\right]\tilde{u}^b(\partial_\phi)^a\,d\widetilde{S}, \tag{132}$$

where the conformal factor is taken to be $\frac{\ell}{r}$.

6.1 Four Dimensional Solutions

In four dimensional case, we find

$$\begin{aligned}
\widetilde{C}^t{}_{rtr} &= \frac{2M\ell^2}{r^5} + O(\frac{1}{r^6}), \\
\widetilde{C}^t{}_{r\phi r} &= \frac{-3Ma\sin^2\theta\ell^2}{\Xi r^5} + O(\frac{1}{r^6}).
\end{aligned} \tag{133}$$

The conformal boundary volume has the form

$$d\widetilde{S}_t = \frac{\ell^2}{\Xi}\sin\theta d\theta d\phi, \tag{134}$$

and

$$\begin{aligned}(\widetilde{E}_1)^t{}_t - (\widetilde{E}_0)^t{}_t &= \Omega^{-1}\left[(C_1)^t{}_{rtr}(\widetilde{n}_1)^r(\widetilde{n}_1)^r - (C_0)^t{}_{rtr}(\widetilde{n}_0)^r(\widetilde{n}_0)^r\right], \\ &\sim \frac{r^5}{\ell^5}\frac{2\ell^2 M}{r^5} = \frac{2M}{\ell^3}.\end{aligned} \tag{135}$$

Thus the conserved charge associated to ∂_t is

$$H_{\partial_t} = \frac{\ell}{8\pi G}\int_{\partial\Sigma}\frac{2M}{\ell^3}\frac{\ell^2}{\Xi}\sin\theta d\theta d\phi = \frac{1}{G}\frac{M}{\Xi}. \tag{136}$$

Similarly we have

$$\begin{aligned}(\widetilde{E}_1)^t{}_\phi - (\widetilde{E}_0)^t{}_\phi &= \Omega^{-1}\left[(C_1)^t{}_{r\phi r}(\widetilde{n}_1)^r(\widetilde{n}_1)^r - (C_0)^t{}_{r\phi r}(\widetilde{n}_0)^r(\widetilde{n}_0)^r\right] \\ &\sim -\frac{r^5}{\ell^5}\frac{-3Ma\sin^2\theta\ell^2}{\Xi r^5} = -\frac{3Ma\sin^2\theta}{\Xi\ell^3},\end{aligned} \tag{137}$$

and

$$H_{\partial_\phi} = \frac{\ell}{8\pi G}\int_{\partial\Sigma}\frac{3m\sin^2\theta}{\Xi\ell^3}\frac{\ell^2}{\Xi}\sin\theta d\theta d\phi = -\frac{1}{G}\frac{Ma}{\Xi^2}. \tag{138}$$

Thus the associated angular momentum is

$$J_\phi = -H_{\partial_\phi} = \frac{1}{G}\frac{Ma}{\Xi^2}. \tag{139}$$

According to the definition [31], the "conformal mass" of the solution is

$$M_c = H_{\partial_t} + \frac{a}{\ell^2}J_\phi = \frac{1}{G}\frac{M}{\Xi^2}, \tag{140}$$

which satisfies the first law of thermodynamics. We note that the mass and angular momentum of the Kerr-Taub-Nut-AdS solution have completely the same form as those of the Kerr-AdS solution.

6.2 Six Dimensional Solutions

For six dimensional solution, we have

$$\begin{aligned}\widetilde{C}^t{}_{rtr} &= \frac{6N\ell^2}{r^6 a\cos\theta} + \frac{12M\ell^2}{r^7} + O(\frac{1}{r^8}), \\ \widetilde{C}^t{}_{r\phi r} &= -\frac{7N\ell^2\sin^2\theta}{\Xi r^6\cos\theta} - \frac{15Ma\sin^2\theta}{\Xi r^7} + O(\frac{1}{r^8}).\end{aligned} \tag{141}$$

The conformal boundary volume takes the form

$$d\widetilde{S}_t = \frac{\ell^4}{\Xi}\cos^2\theta\sin\theta d\theta d\phi d\Omega_2, \tag{142}$$

and

$$\begin{aligned}
(\widetilde{E}_1)^t{}_t - (\widetilde{E}_0)^t{}_t &= \frac{1}{3}\Omega^{-3}\left[(C_1)^t{}_{rtr}(\widetilde{n}_1)^r(\widetilde{n}_1)^r - (C_0)^t{}_{rtr}(\widetilde{n}_0)^r(\widetilde{n}_0)^r\right] \\
&\sim \frac{r^7}{\ell^7}\frac{4\ell^2 M}{r^7} = \frac{4M}{\ell^5}.
\end{aligned} \tag{143}$$

Thus we obtain the conserved charge associated to ∂_t

$$H_{\partial_t} = \frac{\ell}{8\pi G}\int_{\partial\Sigma}\frac{4M}{\ell^5}\frac{\ell^4}{\Xi}\cos^2\theta\sin\theta d\theta d\phi d\Omega_2 = \frac{1}{G}\frac{8\pi M}{3\Xi}. \tag{144}$$

Similarly we find

$$\begin{aligned}
(\widetilde{E}_1)^t{}_\phi - (\widetilde{E}_0)^t{}_\phi &= \frac{1}{3}\Omega^{-3}\left[(C_1)^t{}_{r\phi r}(\widetilde{n}_1)^r(\widetilde{n}_1)^r - (C_0)^t{}_{r\phi r}(\widetilde{n}_0)^r(\widetilde{n}_0)^r\right] \\
&\sim -\frac{r^7}{\ell^7}\frac{5Ma\sin^2\theta\ell^2}{\Xi\ \ r^7} = -\frac{5Ma\sin^2\theta}{\Xi\ell^5},
\end{aligned} \tag{145}$$

$$H_{\partial_\phi} = \frac{\ell}{8\pi G}\int_{\partial\Sigma}\frac{5Ma\sin^2\theta}{\Xi\ell^5}\frac{\ell^4}{\Xi}\cos^2\theta\sin\theta d\theta d\phi d\Omega_2 = -\frac{1}{G}\frac{4\pi Ma}{3\Xi^2}. \tag{146}$$

The associated angular momentum is then given by

$$J_\phi = -H_{\partial_\phi} = \frac{1}{G}\frac{4\pi Ma}{3\Xi^2}. \tag{147}$$

and the "conformal mass" of the solution is

$$M_c = H_{\partial_t} + \frac{a}{\ell^2}J_\phi = \frac{4\pi M}{3G\Xi}(1+\frac{1}{\Xi}). \tag{148}$$

6.3 Eight Dimensional Solutions

In this case we have

$$\begin{aligned}
\widetilde{C}^t{}_{rtr} &= \frac{6N\ell^2}{r^6 a^3\cos^3\theta} - \frac{N\ell^2(a^2+6\ell^2+19a^2\cos^2\theta)}{r^8 a^3\cos^3\theta} + \frac{30M\ell^2}{r^9} + O(\frac{1}{r^{10}}), \\
\widetilde{C}^t{}_{r\phi r} &= -\frac{5N\ell^2\sin^2\theta}{r^6 a^2\cos^3\theta\Xi} + \frac{N\ell^2(5\ell^2+22a^2\cos^2\theta)}{r^8 a^2\cos^3\theta\Xi} - \frac{35Ma\ell^2\sin^2\theta}{r^9\Xi} + O(\frac{1}{r^{10}}),
\end{aligned} \tag{149}$$

and

$$\begin{aligned}
(\widetilde{E}_1)^t{}_t - (\widetilde{E}_0)^t{}_t &= \frac{1}{5}\Omega^{-5}\left[(C_1)^t{}_{rtr}(\widetilde{n}_1)^r(\widetilde{n}_1)^r - (C_0)^t{}_{rtr}(\widetilde{n}_0)^r(\widetilde{n}_0)^r\right] \\
&\sim \frac{r^9}{\ell^9}\frac{6\ell^2 M}{r^9} = \frac{6M}{\ell^7}.
\end{aligned} \tag{150}$$

Note that the conformal boundary volume has the form

$$d\widetilde{S}_t = \frac{\ell^6}{\Xi}\cos^4\theta\sin\theta d\theta d\phi d\Omega_4. \tag{151}$$

We obtain the conserved charge associated to ∂_t

$$H_{\partial_t} = \frac{\ell}{8\pi G}\int_{\partial\Sigma}\frac{6M}{\ell^7}\frac{\ell^6}{\Xi}\cos^4\theta\sin\theta d\theta d\phi d\Omega_4 = \frac{1}{G}\frac{8\pi^2 M}{5\Xi}. \tag{152}$$

Similarly we find

$$\begin{aligned}(\widetilde{E}_1)^t{}_\phi - (\widetilde{E}_0)^t{}_\phi &= \frac{1}{5}\Omega^{-5}\left[(C_1)^t{}_{r\phi r}(\widetilde{n}_1)^r(\widetilde{n}_1)^r - (C_0)^t{}_{r\phi r}(\widetilde{n}_0)^r(\widetilde{n}_0)^r\right] \\ &\sim -\frac{r^9}{\ell^9}\frac{7Ma\sin^2\theta\ell^2}{\Xi\ \ r^9} = -\frac{7Ma\sin^2\theta}{\Xi\ell^7},\end{aligned} \tag{153}$$

$$H_{\partial_\phi} = \frac{\ell}{8\pi G}\int_{\partial\Sigma}\frac{7Ma\sin^2\theta}{\Xi\ell^7}\frac{\ell^6}{\Xi}\cos^4\theta\sin^3\theta d\theta d\phi d\Omega_4 = -\frac{1}{G}\frac{8\pi^2 Ma}{15\Xi^2}. \tag{154}$$

Thus the angular momentum is

$$J_\phi = -H_{\partial_\phi} = \frac{1}{G}\frac{8\pi^2 Ma}{15\Xi^2}, \tag{155}$$

and the "conformal mass" of the solution

$$M_c = H_{\partial_t} + \frac{a}{\ell^2}J_\phi = \frac{8\pi^2 M}{15G\Xi}(2+\frac{1}{\Xi}). \tag{156}$$

6.4 Ten Dimensional Solutions

This case gives

$$\begin{aligned}\widetilde{C}^t{}_{rtr} &= \frac{6N\ell^2}{r^6a^5\cos^5\theta} - \frac{N\ell^2\left(6\ell^2+3a^2+17a^2\cos^2\theta\right)}{r^8a^5\cos^5\theta}, \\ &+ \frac{N\ell^2\left(3\left(a^4+a^2\ell^2+2\ell^4\right)-a^2\left(a^2-17\ell^2\right)\cos^2\theta+40a^4\cos^4\theta\right)}{r^{10}a^5\cos^5\theta} \\ &+ \frac{56M\ell^2}{r^{11}} + O(\frac{1}{r^{12}}) \\ \widetilde{C}^t{}_{r\phi r} &= -\frac{3N\ell^2\sin^2\theta}{r^6a^4\cos^5\theta\Xi} + \frac{3N\ell^2(\ell^2+6a^2\cos^2\theta)\sin^2\theta}{r^8a^4\cos^5\theta\Xi} \\ &- \frac{3N\ell^2(\ell^4+6a^2\ell^2\cos^2\theta+15a^2\cos^4\theta)\sin^2\theta}{r^{10}a^4\cos^5\theta\Xi} - \frac{63Ma\ell^2\sin^2\theta}{r^{11}\Xi} + O(\frac{1}{r^{12}}).\end{aligned} \tag{157}$$

The conformal boundary volume has the form

$$d\widetilde{S}_t = \frac{\ell^8}{\Xi}\cos^6\theta\sin\theta d\theta d\phi d\Omega_6, \tag{158}$$

Note that

$$\begin{aligned}(\widetilde{E}_1)^t{}_t - (\widetilde{E}_0)^t{}_t &= \frac{1}{7}\Omega^{-7}\left[(C_1)^t{}_{rtr}(\widetilde{n}_1)^r(\widetilde{n}_1)^r - (C_0)^t{}_{rtr}(\widetilde{n}_0)^r(\widetilde{n}_0)^r\right] \\ &\sim \frac{r^{11}}{\ell^{11}}\frac{8\ell^2 M}{r^{11}} = \frac{8M}{\ell^9}.\end{aligned} \tag{159}$$

We find the conserved charge associated to the Killing vector ∂_t

$$H_{\partial_t} = \frac{\ell}{8\pi G}\int_{\partial\Sigma} \frac{8M}{\ell^9}\frac{\ell^8}{\Xi}\cos^6\theta\sin\theta d\theta d\phi d\Omega_6 = \frac{1}{G}\frac{64\pi^3 M}{105\Xi}. \tag{160}$$

Similarly one can get

$$\begin{aligned}(\widetilde{E}_1)^t{}_\phi - (\widetilde{E}_0)^t{}_\phi &= \frac{1}{7}\Omega^{-7}\Big[(C_1)^t{}_{r\phi r}(\widetilde{n}_1)^r(\widetilde{n}_1)^r - (C_0)^t{}_{r\phi r}(\widetilde{n}_0)^r(\widetilde{n}_0)^r\Big] \\ &\sim -\frac{r^{11}}{\ell^{11}}\frac{9Ma\sin^2\theta\ell^2}{\Xi\ \ r^{11}} = -\frac{9Ma\sin^2\theta}{\Xi\ell^9},\end{aligned} \tag{161}$$

$$H_{\partial_\phi} = \frac{\ell}{8\pi G}\int_{\partial\Sigma} \frac{9Ma\sin^2\theta}{\Xi\ell^9}\frac{\ell^8}{\Xi}\cos^6\theta\sin^3\theta d\theta d\phi d\Omega_6 = -\frac{1}{G}\frac{16\pi^3 Ma}{105\Xi^2}. \tag{162}$$

Thus the solution has the angular momentum

$$J_\phi = -H_{\partial_\phi} = \frac{1}{G}\frac{16\pi^3 Ma}{105\Xi^2}, \tag{163}$$

and the "conformal mass"

$$M_c = H_{\partial_t} + \frac{a}{\ell^2}J_\phi = \frac{16\pi^3 M}{105G\Xi}(3+\frac{1}{\Xi}). \tag{164}$$

7 (Un)Wrapped Brane

The black brane solutions with flat transverse space in n-dimensions are also AlAdS space-times. Their metrics can be written as [35, 36]

$$ds^2 = -\Delta(r)^2 dt^2 + \frac{dr^2}{\Delta(r)^2} + r^2({dx_1}^2 + \cdots + {dx_{n-2}}^2) \tag{165}$$

where $\Delta(r)^2 = -2M/r^{n-3} + r^2/\ell^2$. In this metric, at least one of the transverse direction x_i should be compactified so that parameter M cannot be changed by rescaling the coordinates. Thus, the non-vanishing conserved charge is the mass only. Choosing the background reference solution as $M = 0$, we have

$$\widetilde{E}^t{}_t \sim \frac{(n-2)M}{\ell^{n-1}}. \tag{166}$$

The boundary conformal volume element has the form

$$d\widetilde{S}_t = \ell^{n-2}dv, \tag{167}$$

where dv denotes the volume element of $n-2$-transverse space. We thus obtain

$$H_{\partial_t} = \frac{\ell}{8\pi G}\int \frac{(n-2)M}{\ell^{n-1}}\ell^{n-2}dv = \frac{(n-2)M}{8\pi G}V, \tag{168}$$

where V represents the volume of $n-2$-transverse space. This result is the same as that of [14, 15]. Note that this solution gives an example which has a Ricci flat boundary, so that the Fefferman-Graham expansion without log-terms can be done in any dimension [26], and the conserved charges can be defined as in (61).

8 Conclusion and Discussion

In this paper, based on the work of Hollands *et al.* [12], we derive a formula of calculating conserved charges in even dimensional asymptotically *locally* Anti-de Sitter (AlAdS) space-times by using the covariant phase space definition of Wald and Zoupas [19]. Our formula generalizes the formula proposed by Ashtekar *et al.* [5, 10]. This formula is background dependent. We therefore have to specify a reference solution when we calculate conserved charges for a certain AlAdS space-time. Using this formula we calculate the masses of Taub-Bolt-AdS space-times by treating Taub-Nut-AdS space-times as reference solutions. The resulting masses agree with those obtained previously by “background subtraction” method and “boundary counterterm” method. We also discussed the conserved charges in four dimensional Kerr-Taub-Nut-AdS solutions and higher dimensional Kerr-AdS solutions with Nut charges by treating the corresponding massless solutions as the background reference solutions. For these higher dimensional Kerr-AdS solutions with Nut charges, these conserved charges are obtained at the first time. In addition, as a further example of AlAdS space-times, the mass of the (un)wrapped brane solutions in any dimensions is also studied.

It is interesting to discuss the odd dimensional case. However, some log terms will appear in the Fefferman-Graham expansion in this case. Therefore it will fail by naively applying the same procedure as is exhibited in this paper to the odd dimensional case. Nevertheless, we have found that the similar conserved charges can be defined for any dimension if the static AlAdS space-times have Ricci flat boundaries.

Topological AdS black holes also belong to a kind of AlAdS space-times, and they may have nontrivial horizons and boundary topologies (see for example [37, 38, 39, 40, 35]). This study is motivated by the discovery of Bañados-Teitelboim-Zanelli (BTZ) black holes [41], which are exact solutions in the three-dimensional Einstein gravity with a negative cosmological constant, and are locally equivalent to a three-dimensional anti-de Sitter space. The method of Ashtekar *et al.* can not be used directly to compute the conserved charges of these black hole space-times. So, it is interesting to discuss the conserved charges of these solutions by using our new formula.

The examples discussed in this paper suggest that in even dimensional AlAdS space-times, a relation of “surface counterterm method” to our method

$$H_\xi[g] = Q_\xi[g] - Q_\xi[g_0], \tag{169}$$

where $Q_\xi[g]$ is the conserved charge obtained by using the “surface conterterm” method for a solution g and $Q_\xi[g_0]$ is the one corresponding to the selected reference solution g_0. If g is an AAdS space-time, this relation reduces to

$$H_\xi[g] = Q_\xi[g] - Q_\xi[AdS]. \tag{170}$$

This case has been discussed by Hollands, Ishibashi and Marolf in the paper [42]. They have used general arguments based on the Peierls bracket to compare the counterterm charges and the Hamiltonian charges defined in [12] in any dimensional AAdS space-time. In the even dimensional AAdS case, the counterterm charge of exact AdS space-times vanishes, while in odd dimensional case the counterterm charge $Q_\xi[AdS]$ corresponds to the Casimir energy of the boundary CFTs. So, it is interesting to find to what the counterterm charges $Q_\xi[g_0]$ correspond in the boundary CFTs as one considers the general AlAdS cases.

Acknowledgements

L.M.Cao thanks Hong-Sheng Zhang, Hao Wei, Hui Li, Da-Wei Pang and Yi Zhang for useful discussions and kind help. This work is supported by grants from NSFC, China (No. 10325525 and No. 90403029), and a grant from the Chinese Academy of Sciences.

References

[1] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1998)] [arXiv:hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998) [arXiv:hep-th/9802109].

[3] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150]; E. Witten, Adv. Theor. Math. Phys. **2**, 505 (1998) [arXiv:hep-th/9803131].

[4] M. Henneaux and C. Teitelboim, Commun. Math. Phys. **98**, 391 (1985).

[5] A. Ashtekar and A.Magnon, Class. Quant. Grav. **1**, L39 (1984).

[6] L. F. Abbott and S. Deser, Nucl. Phys. B **195**, 76 (1982).

[7] S. W. Hawking and G. T. Horowitz, Class. Quant. Grav. **13** (1996) 1487 [arXiv:gr-qc/9501014].

[8] J.D. Brown and J.W. York, Phys. Rev. **D47**, 1407 (1993).

[9] V. Balasubramanian and P. Kraus, Commun. Math. Phys. **208**, 413 (1999) [arXiv:hep-th/9902121].

[10] A. Ashtekar, S. Das, Class. Quant. Grav. **17** (2000) L17-L30 [arXiv:hep-th/9911230].

[11] R. Olea, JHEP **0506**, 023 (2005) [arXiv:hep-th/0504233].

[12] S. Hollands, A. Ishibashi, and D. Marolf, Class. Quant. Grav. **22** (2005) 2881-2920 [arXiv:hep-th/0503045].

[13] I. Papadimitriou and K. Skenderis, JHEP **08** (2005) 004 [arXiv:hep-th/0505190].

[14] R. Aros, M. Contreras, R. Olea, R. Troncoso, and J. Zanelli, Phys. Rev. Lett. **84** (2000) 1647-1650 [arXiv: gr-qc/9909015].

[15] R. Aros, M. Contreras, R. Olea, R. Troncoso, and J. Zanelli, Phys. Rev. D **62** (2000) 044002 [arXiv: hep-th/9912045].

[16] N. Deruelle and J. Katz, Class. Quant. Grav. **23**, 753 (2006) [arXiv:gr-qc/0512077].

[17] R. M. Wald and A. Zoupas, Phys. Rev. D **61**(2000) 084027 [arXiv:gr-qc/9911095].

[18] J. Lee and R. M. Wald, J. Math. Phys. **31**(3) 725 March 1990.

[19] V. Iyer and R. M. Wald, Phys. Rev. D **50** (1994) 846-864 [arXiv:gr-qc/9403028].

[20] V. Iyer and R. M. Wald, Phys. Rev. D **52** (1995) 4430-4439[arXiv:gr-qc/9503052].

[21] C. R. Graham [arXiv:math.DG/9909042].

[22] Michael T. Anderson [arXiv: hep-th/0403087].

[23] S. de Haro, K. Skenderis, and S. N. Solodukhin, Commun. Math. Phys. **217** (2001) 595-622 [arXiv:hep-th/0002230].

[24] M. Henningson and K. Skenderis, JHEP **9807** (1998) 023 [hep-th/9806087]; M. Henningson and K. Skenderis, Fortsch. Phys. **48** (2000) 125 [hep-th/9812032]; S. Nojiri and S. D. Odintsov, Phys. Lett. B **444**, 92 (1998) [arXiv:hep-th/9810008].

[25] C. Fefferman and C. Robin Graham, "Conformal Invariants", in *Elie Cartan et les Mathématiques d'aujourd'hui* (Astérisque, 1985) 95.

[26] G.J. Galloway, S. Surya, and E. Woolgar, Commun. Math. Phys. **241** (2003) 1-25[arXiv: hep-th/0204081].

[27] A. Awad and A. Chamblin Class. Quant. Grav. **19** (2002) 2051-2062[arXiv: hep-th/0012240].

[28] S.W. Hawking, C.J. Hunter, D.N. Page Phys. Rev. D **59** (1999) 044033[arXiv: hep-th/9809035].

[29] A. Chamblin, R. Emparan, C. V. Johnson, and R. C. Myers Phys. Rev. D **59** (1999) 064010 [arXiv: hep-th/9808177].

[30] R. Clarkson, L. Fatibene, and R.B. Mann, Nucl. Phys. B **652** (2003) 348-382[arXiv: hep-th/0210280]; D. Astefanesei, R. B. Mann and E. Radu, Phys. Lett. B **620**, 1 (2005) [arXiv:hep-th/0406050]; D. Astefanesei, R. B. Mann and E. Radu, JHEP **0501**, 049 (2005) [arXiv:hep-th/0407110].

[31] G.W.Gibbons, M.J.Perry, and C.N. Pope, Class. Quant. Grav. **22** (2005) 1503-1526[arXiv:hep-th/0408217].

[32] R.B. Mann, Phys. Rev. D **61** (2000) 084013 [arXiv: hep-th/9904148].

[33] Z.-W. Chong, G.W. Gibbons, H. Lu, and C.N. Pope, Phys. Lett. B **609** (2005) 124-132[arXiv: hep-th/0405061].

[34] S.W.Hawking, C.J.Hunter, and M.M.Taylor-Robinson, Phys. Rev. D **59** (1999) 064005[arXiv: hep-th/9811056].

[35] D. Birmingham, Class. Quant. Grav. **16**, 1197 (1999) [arXiv:hep-th/9808032]; R. G. Cai and K. S. Soh, Phys. Rev. D **59**, 044013 (1999) [arXiv:gr-qc/9808067].

[36] G. T. Horowitz and R. C. Myers, Phys. Rev. D **59** (1999) 026005[arXiv: hep-th/9808079]

[37] S. Aminneborg, I. Bengtsson, S. Holst, and P. Peldan, Class. Quantum. Grav. **13** 2707 (1996); M. Bañados, Phys. Rev. D **57** 1068 (1998); M. Bañados, A. Gomberoff, and C. Martínez, hep-th/9805087.

[38] J. P. S. Lemos, Phys. Lett. B **353**, 46 (1995); Class. Quantum Grav. **12**, 1081 (1995); J. P. S. Lemos and V. T. Zanchin, Phys. Rev. D **54** 3840 (1996); C. G. Huang and C. B. Liang, Phys. Lett. A **201**, 27 (1995).

[39] R. G. Cai and Y. Z. Zhang, Phys. Rev. D **54**, 4891 (1996) [arXiv:gr-qc/9609065]; R. G. Cai, J. Y. Ji and K. S. Soh, Phys. Rev. D **57**, 6547 (1998) [arXiv:gr-qc/9708063].

[40] R. B. Mann, Class. Quant. Grav. **14**, L109 (1997) [arXiv:gr-qc/9607071]; D. R. Brill, J. Louko and P. Peldan, Phys. Rev. D **56**, 3600 (1997) [arXiv:gr-qc/9705012]; L. Vanzo, Phys. Rev. D **56**, 6475 (1997) [arXiv:gr-qc/9705004].

[41] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992); M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. D **48**, 1506 (1993).

[42] S. Hollands, A. Ishibashi, and D. Marolf, Phys. Rev. D **72**(2005) 104025 [arXiv:hep-th/0503105].